

THE SECOND YEARBOOK OF THE NATIONAL COUNCIL

The Second Yearbook of the National Council of Teachers of Mathematics will be devoted to a consideration of "Curriculum Problems in Teaching Mathematics." The discussion will be divided into three parts as follows:

Part I will be devoted to "The Teaching of Arithmetic." The contributors for this section are Professors F. B. Knight of the University of Iowa and G. T. Buswell of the University of Chicago, and Miss Jessie P. Haynes, Supervisor of Mathematics in Richmond, Virginia.

Part II on "The Teaching of Mathematics in the Junior High School" will be discussed by Professor Ralph Beatley of Harvard University, Mr. Harry C. Barber of Charlestown High School, Boston, Massachusetts, Mr. C. Louis Thiele, Assistant Director of Exact Sciences, Detroit, Michigan, and Professors David Eugene Smith and W. D. Reeve of Teachers College, Columbia University.

Part III will be devoted to "The Teaching of Mathematics in the Senior High School." The contributors to this section are Miss Gertrude Allen of the University High School, Oakland, California, Mr. Eugene R. Smith of the Beaver Country Day School, Chestnut Hill, Massachusetts, and Professor David Eugene Smith of Teachers College, Columbia University.

The Yearbook will probably be ready for distribution by February 15th and can be secured on or after that date by sending \$1.25 to The Bureau of Publications, Teachers College, Columbia University.

W. D. REEVE
C. N. STOKES
C. B. MARQUAND

1927 Yearbook Committee

The Slide Rule

in Mathematics and Trigonometry



NO course in Mathematics or Trigonometry can be called complete if it does not include instruction in the use of the Slide Rule. It provides many short cuts in mathematical calculations and has proved an efficient check in trigonometry.

Let your pupils enjoy and profit from a series of lessons in the use of the Slide Rule. Ask about our

Demonstration Slide Rule

furnished at nominal price for classroom use.

K & E Slide Rules

are used almost exclusively at leading institutions of learning. They have an established reputation for fine quality and accuracy. Our manuals make self-instruction easy for teacher and pupil.



Personal direction and instruction in the use of the slide rule may be secured by writing to the Home Study Department of Columbia University, New York City.

KEUFFEL & ESSER CO.

NEW YORK, 127 Fulton Street,

General Office and Factories, HOBOKEN, N. J.

CHICAGO

ST. LOUIS

SAN FRANCISCO

MONTREAL

223-225 Dearborn St.

517 Locust St.

20-24 Second St.

6 Notre Dame St. W.

Selling Materials, Mathematical and Surveying Instruments, Measuring Tapes

Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS
IN JUNIOR AND SENIOR HIGH SCHOOLS

VOLUME XX

FEBRUARY, 1927

NUMBER 2

General Mathematics	RALEIGH SCHORLING	65
Rigor Versus Expediency in the Proof of Locus Originals	ELMER R. BOWEN	82
Proof of an Original Exercise	WALTER HEYER	91
The Arithmetical Productiveness of Unsharian, Social and Scientific Ideals: Viewed Historically	G. W. MYERS	93
The Use of Problems in Teaching Elementary Algebra	R. H. TAYLOR	101
The Middle of the Road	W. A. SAYDER	112
National Council Program		125
The Second Yearbook of the National Council of Teachers of Mathematics		126

Published by the

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

LANCASTER, PA.

NEW YORK

Entered as second-class matter, November 10, 1925, at the Post Office at Lancaster, N. Y., under the Act of March 3, 1879. Accepted for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized November 17, 1925. Application for transfer to Lancaster, Pa., postoffice, pending.

THE MATHEMATICS TEACHER

THE OFFICIAL JOURNAL OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Edited by

R. G. CLARK, Editor-in-Chief

DAVID L. DAVIS, Associate Editor

DAVID L. DAVIS

D. GAVIN

A. FOSTER

MARY GUILD

W. W. YOUNG

The Council has an Advisory Board consisting of

W. W. YOUNG, Chairman

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, W. W. YOUNG

W. W. YOUNG, A. L. LUBY

OSCAR W. MATHIAS

JOHN H. MATHIAS

W. D. RAYNE

RAYMOND SCHMIDT

HARRY M. KRAL

RAYMOND M. WILSON

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

DAVID L. DAVIS, Assistant Superintendent of Schools,

Cleveland, Ohio

W. W. YOUNG, University of Wisconsin

Madison, Wisconsin

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

W. W. YOUNG, University of Illinois (1907)

Urbana, Illinois (1907)

PRICES OF REPRINTS

100	200	300	400	500	600	700	800	900	1000
1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50
6.00	9.00	12.00	15.00	18.00	21.00	24.00	27.00	30.00	33.00
36.00	54.00	72.00	90.00	108.00	126.00	144.00	162.00	180.00	198.00
216.00	324.00	432.00	540.00	648.00	756.00	864.00	972.00	1080.00	1188.00

Prices include postage and handling charges.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

For 100 reprints, \$1.00; for 200, \$1.50; for 300, \$2.00; for 400, \$2.50; for 500, \$3.00; for 600, \$3.50; for 700, \$4.00; for 800, \$4.50; for 900, \$5.00; for 1000, \$5.50.

THE MATHEMATICS TEACHER

VOLUME XX

FEBRUARY, 1927

NUMBER 2

GENERAL MATHEMATICS

BY PROFESSOR RALEIGH SCHORLING,

University of Michigan

A Significant Change.—Twenty-five years ago there were, so far as is known, no students studying courses in general mathematics. To-day the pupils enrolled in courses in general mathematics are numbered by the hundred thousands. It must, therefore, be admitted that the rapid growth of general mathematics constitutes one of the striking changes in the first quarter of this century.

What is General Mathematics?—General mathematics is an introductory, basic, exploratory course in which the simple and significant principles of arithmetic, algebra, intuitive geometry, statistics, and numerical trigonometry are taught so as to emphasize their natural and numerous interrelations.

The movement started about ten years ago and represented an effort to get a course in the ninth year which would more nearly meet the needs of pupils, particularly those of low ability and poor background and those who would leave school early. There was no intention to set up three or four years of college preparatory courses. The emphasis was on the notion of a basic, exploratory *one year* course. For evidence the reader may turn to the preface of the earliest ¹ books written on general mathematics. At that time it was not possible to teach such a course in the seventh and eighth grades to any considerable extent because secondary school people did not control the seventh and eighth grades.

It is important to note the distinction between general mathematics, correlated (or fused) mathematics, and parallel treatment. Correlated mathematics puts the emphasis upon correlations. Its best example is to be found in the mathematics

¹ See *General Mathematics*, Schorling-Reeve; *Fundamentals of High School Mathematics*, Rugg-Clark.

taught in the University High School at the University of Chicago. This course is the outcome of Professor Moore's plea in his presidential address² for organizing "the algebra, geometry, and physics of the Secondary School into a thoroughly coherent four years' course, comparable in strength and closeness of structure with the four years' course in Latin." Professors Myers and Breslich are largely responsible for the effort at the University of Chicago to construct a four years' course in correlated mathematics. It should be pointed out that they did not attempt to incorporate in their program the notion of an introductory course which would "skim the cream" off the various high school subjects for beginners and those of low ability. A second distinction between correlated mathematics is a more extreme emphasis on the interrelations in so far as plane geometry is broken up and the units distributed among the other mathematics subjects of the high school. That is to say, there is nowhere a year or a half year of demonstrative geometry as such in the Chicago program. To many mathematicians this is an important and indeed a vital issue.

Parallel treatment, as its name implies, refers to the organization of mathematics in which, on certain days of the week pupils study a given subject, let us say algebra, and on the remaining days study another, as for example geometry. By this method arithmetic and geometry (or some other combination of subjects) are given side by side. Their interrelationships are pointed out from time to time but there is no effort to merge the subject matter of two or more branches of mathematics. This treatment is very commonly used in European schools and has a time or two been given a trial in certain city schools of America.

— *What Were Some of the Causes That Gave Rise to General Mathematics?*—The causes may be conveniently classified under two headings: (1) the new philosophy of secondary education, (2) the status of mathematics a decade ago.

A New Philosophy. In the decade which preceded the organization of general mathematics, changing social conditions crowded girls and boys into the secondary schools with less

² See Professor E. H. Moore's *Presidential Address* in the *First Year-book of the National Council of Teachers of Mathematics*, page 47.

native ability and with far less social experience as background to do the work in mathematics successfully. School men began to realize the very low degree of mastery exhibited even by pupils with passing marks. The materials in mathematics were "geared" too high for large groups of pupils. Moreover, many essentials in mathematics of considerable social value were so long delayed that many pupils dropped out of school before they had an opportunity to profit by them. There was the conviction that mathematics must yield its organization to meet the needs of the pupils. This philosophy was probably most adequately expressed in "Cardinal Principles of Secondary Education." To be sure the early books in general mathematics antedated this document but the philosophy was "in the air," and was directly responsible for general mathematics.

The Status of Mathematics Ten Years Ago.—It is possible and indeed probable that the reorganization movement in mathematics would have come without the drive of this new philosophy. For there was ten years ago a keen dissatisfaction among mathematicians with the organization of the mathematics in our high schools. Leaders in this country were beginning to see in the works of the German writers Klein and Tropfke the implication that possibly we were missing the very heart of mathematics, namely, the appreciation of the power of mathematics. We failed to see the rôle that mathematics has played in the development of civilization, in particular, in science and in industry. They were questioning the value of the excessive formal manipulations in algebra largely inherited from Wentworth. Some asserted that pupils were not being "let in on the secret" as to what it was all about.

That there was great dissatisfaction even among conservative mathematicians is shown by the following memorandum which is probably the severest and most startling criticism ever made of the teaching of mathematics in our secondary schools. A few of the most significant statements from the memorandum³ are given here.

"The situation that needs to be met may best be illustrated by the case of algebra. Our elementary

³ This refers to a memorandum written by a committee representing the Mathematical Association of America and addressed to the General Education Board. The memorandum secured generous funds for the National Committee on Mathematical Requirements.

algebra is, in theory and symbolism, substantially what it was in the seventeenth century. The present standards of drill work, largely on non-essentials, were set up about fifty years ago. The few lines of application of algebra to nominally practical questions found in our college entrance examinations are mere variants of problems that are centuries old, and that often represent only remotely any real condition of to-day. A considerable number of teachers, both in the Secondary Schools and the colleges, believe that the amount of time spent by pupils on abstract work in difficult problems, in division, factoring, fractions, simultaneous equations, radicals, etc., is excessive; that such work leads to nothing important in the science, and adds but little to facility in the manipulation of algebraic forms.

"It is urged by many friendly critics that instead of giving the students a good all-round idea of what mathematics means and its general range of application our present secondary school courses are too abstract, often uninteresting, except to the mathematically inclined, and not as valuable as they might be as an aid to college in general, or to life work for those who enter at once into their careers."

Bear in mind that the statement is signed by all who were members of the committee at the time the document was issued and that all members were teachers of mathematics, representing officially the large mathematical organizations—collegiate and secondary—of the country. No further evidence is necessary; the case against mathematics was frankly admitted by those presumably competent to judge.

A second influence at that time was the Perry movement in England. About thirty-five years ago, Perry was strongly emphasizing the practical sides of mathematics and with lessened emphasis on the systematic and formal organization of the subject. Perry insisted that it was entirely logical in the earlier years to take a large list of basic principles rather than a smaller one, and build a structure of mathematics on this larger list, thus reserving for the later years the criticism of this basic list of principles and by means of this criticism reducing the number

of basic theorems. The moment this is admitted, and it has been definitely accepted by all leading men in mathematics,⁴ the door is wide open for a course in general mathematics. The Perry movement was given emphasis in this country among mathematicians by President Moore's brilliant presidential address at the American Mathematical Society (1902). In this address Professor Moore made his plea for a coherent four years' course, to which reference has already been made. In the years immediately following, Professor George Myers made a genuine attempt to break up the "watertight compartment" method of teaching algebra, arithmetic, geometry and trigonometry. While the chief objective of general mathematics has been upon the needs of common life, nevertheless the correlated movement paved the way in that it represented an extreme and vigorous attack on the formal organization of logical units of mathematics commonly thrown at children.

Junior High School Books.—Another factor contributing greatly to preparing the way for general mathematics was the appearance of junior high school texts in mathematics written by conservative writers⁵ in which no one year was devoted to a single school subject. For example, in the first series appearing, for the junior high school, the seventh grade book was broken into two parts—the first half devoted to arithmetic and the last half to algebra; the first half of the eighth grade book was given to algebra and the last half to arithmetic; and the first half of the ninth year book consisted of algebra and the last half of a unit of demonstrative geometry. Obviously, this was an important change. If it is admitted that a year's work in junior high school grade ought to include something from each of two or more branches, it's an easy step to the conviction that it's more sensible to teach the various branches in a way that will emphasize their natural interrelationships; hence, again, general mathematics.

The Writings of T. Percy Nunn.—The brilliant work of Professor Nunn in America is but little known but it has clearly changed our procedures. Just now the effect is being felt in the teaching of English. Pedagogical writers in America have only

⁴ See President Moore's *Address*, First Yearbook, National Council of Teachers of Mathematics, page 47.

⁵ See Wentworth-Smith-Brown, *Junior High School Mathematics*, Volumes 1, 2, 3 (1917).

recently discovered Nunn, but books on mathematics for high school pupils began to show his influence about ten years ago. Nunn is unique in that possibly he was the first mathematician to recognize the importance of psychologizing the subject of mathematics. Though regarded as a rigorous mathematician, Nunn is never concerned about the logical organization of a unit of subject-matter for children so long as the organization is one that contributes to ease in learning. In his writing he exhibits a marvelous insight for the ways children master the introductory concepts of mathematics.

By way of summary, ten years ago the following forces were operating in the field of mathematics so as to give rise to the general mathematics movement:

- (1) A new philosophy of secondary education.
- (2) Dissatisfaction with the organization of high school mathematics expressed by teachers of college mathematics.
- (3) The writings of leaders in Germany.
- (4) The Perry movement in England.
- (5) The influence of Professor Moore's address and the correlated idea promoted at the University of Chicago.
- (6) The appearance of junior high school texts.
- (7) The writings of T. Percy Nunn suggesting the possibilities in the application of psychology to the teaching of mathematics.

What Are the Aims of a Course in General Mathematics?—

If one glances through the literature of general mathematics, it is noted that the following outcomes were expected as the results of organizing such courses:

On the side of subject matter:

- (1) Greater facility in the use of fractions (common and decimal) and of percentage relations.
- (2) An introduction to trigonometry and statistics probably adequate for common needs.
- (3) Control of the simple and important parts of algebra and geometry.
- (4) Training in the use of a number of "optional" topics, e.g., logarithms, the slide-rule and tables.

As concerns the pupil:

- (1) The considerable reduction of the number of first year failures.
- (2) Increase in the number of pupils taking a subsequent course in plane geometry with interest and profit.
- (3) More intelligent election of later courses in mathematics.
- (4) More adequate preparation for the mathematical needs of other school subjects, industrial arts, household arts, physics, chemistry and the like.
- (5) A beginning on the part of the student in the technique of investigation (many parts of the material are organized in laboratory form).
- (6) Greater power in problem solving through the use of more methods of attack (correlation is used as a means and not as an end).
- (7) A clearer notion of the relationship of various mathematical methods without a forced correlation (there is no attempt to correlate plane geometry with material from other fields).
- (8) A better understanding of algebra as far as it goes (this is probably well within the limits of ordinary life needs).
- (9) An appreciation and understanding of the importance of the idea of relationship (function concept).
- (10) Greater enjoyment in the study of mathematics.

What Are the Conditions Favoring the Growth of General Mathematics?—Among the factors now operating toward the extension of general mathematics are the following:

(1) The population of our high schools has in the last quarter of a century been multiplied by more than ten whereas the population of our nation has not doubled. This increase in school population has given us a wider sampling of the general public and hence has in all probability lowered the level of ability. Certainly it has given us pupils with less background to do the conventional courses successfully. The language difficulties which the teacher confronts in instructing the children of recent immigrants—a problem met in many high schools—is alone very great.

(2) The number of children who should take one-year courses in mathematics is very large. There are cities of considerable size in Michigan in which eighty percent of the children still

drop out of school before entering senior high school. The mathematical materials that should be emphasized are computation and the simple principles in algebra and geometry which they will need in general reading, in the shops, and in the commercial pursuits. Considering the large groups of pupils who have a serious need for a "flunkers' bonanza," it is surprising the few books that have been published to meet this demand.⁶ It is to be hoped that writers will interest themselves in producing materials adjusted for this important group in our secondary schools.

(3) The investigation conducted by the National Committee on Mathematical Requirements and reported on page 45 in the *Reorganization of Mathematics in Secondary Education* indicates clearly that what the college man or woman needs to know are precisely those elementary principles which a half dozen series of junior high school textbooks are striving to teach with great emphasis. In the report of the Committee we read:

"It is interesting to note how closely the modifications suggested by this inquiry (on college needs) correspond to the modifications in secondary school mathematics foreshadowed by the study of needs of the high school pupil irrespective of his possible future college attendance."

and later we read:

"That they should be in such close accord with the desires of college teachers in the fields of physical and social sciences as to entrance requirements is striking."

(4) The rapid growth of the junior high school movement has given teachers a greater opportunity to teach worthwhile courses in the seventh and eighth grades. To be sure there is no reason why the same course could not be given by a competent teacher in the seventh and eighth grades of the conventional elementary school. But the fact is that administrators are more anxious to

⁶ So far as the writer knows there are only four books which have been written with a definite purpose in mind to get a one-year course for the less able students. These are *Modern Mathematics—Briefer Course*, by Schorling-Clark-Rugg, replacing the earlier volume by Rugg and Clark, *Fundamentals of High School Mathematics*; *General Mathematics*, by Smith-Foberg-Reeve, replacing an earlier volume—*General Mathematics* by Schorling-Reeve.

initiate new and substantial work in these grades once they have accepted the machinery of the junior high school.

(5) The large number of failures, together with the very low mastery on the part of pupils who pass the courses, makes it necessary that we organize the materials in the form in which they are most readily learned. General mathematics tries to utilize a wider range of sensory experience. Everything else being equal, a problem accompanied by a graphic picture is more easily understood and is appreciated by a greater number of pupils. Can "easy" mathematics be worth while? The psychologist says that a subject cannot be made too easy. Surely the teacher who by keener insight into the nature of mental life succeeds in making mathematics clear and vivid is to be preferred to the teacher who by lack of psychological insight presents the same subject matter in a way that makes it difficult for pupils. There is evidence that a large number of pupils in our high schools do not have the ability necessary to pass the formal algebra course,—at any rate not without some preparatory course along the lines of general mathematics in grades seven and eight.

(6) *General mathematics facilitates motivation.* There is greater opportunity for richer application throughout the junior high school years because the logical order has been replaced by the psychological order. The pupil learns the important principles of algebra and geometry much earlier and along with his arithmetic. If a pupil becomes interested in a problem in another school subject or in his out-of-school experience, he need not be told that he must wait three or four years before he can understand its solution. Since the materials are drawn from several different fields, the illustrations are more varied. Hence the general courses are believed to be more interesting for most pupils.

The development of the junior high school has encouraged the classroom teachers to think more carefully about the problem of purpose and the control of attention. Perhaps they are forced to do so. Certainly it is inconceivable that a teacher could teach five or six large classes of buoyant youth, in the junior high school with the methods commonly used in college.

Search through the literature of interest, purpose, and motivation nowhere yields a systematic treatment of the control

of attention. Must every problem be, as Merriam says, a "self-starter?" Is there anything a teacher can do or dare do the "morning-after" when the machine is cold? Does the psychology of interest offer a few definite practical guides to the classroom teacher for the control of attention or is the whole matter of our responses in the classroom that of chance? There are those of us who believe that the reactions of individual pupils form a resultant quite as definite as that obtained by the law of the parallelogram of forces in physics. With this point of view, the following outline is submitted to teachers of the junior high school with the hope that it may serve as a brief guide in the control of attention.

THE PSYCHOLOGY OF INTEREST⁷

I. Annoyance.—*The first condition of interest is the absence of mental and physical annoyance.* This principle is often neglected even by teachers of wide experience.

II. Enthusiasm.—*The teacher must exhibit or simulate enthusiasm for and joy in her work.* Suffuse the period with good cheer and sociability.

III. Understanding.—*Understanding is a condition of interest.* Material must be presented in language written not only for the pupil but to the pupil. It must be vividly illustrated by human interest stories related to pupil's experiences, sketches, cartoons, graphs.

IV. Variety.—*To maintain interest an activity must be characterized by a change of pace, of materials, of methods, of physical and mental environment.* Variety is probably the most desired attribute of a good lesson.

V. Sensory Experience.—*Everything else being equal, pupils will be more interested if the materials are taught in such a way as to utilize extensive sensory experiences.* There must be provision for the graphic, for the dramatic, for manipulation and the like.

VI. Rate.—*Everything else being equal, the material should be developed at a rapid rate.* This probably has application even for slow pupils. Not even pupils of low intelligence have no trouble in following a rapid rate. The writer has frequently in supervision improved the

⁷ *Definition.*—Interest is here defined as that driving force which enables a pupil to undertake a project and which enables him to

of a class by breaking the work into smaller units and speeding the rate.

VII. *Problem Solving.*—Fundamentally the mind is interested in seeking and discovering relationships. Life is largely made up of problems. Know Dewey's formula. Give emphasis to laboratory methods, to wide experience and to generalization by pupil. Use inductive methods and puzzles. Get the spirit and method of research even at the elementary school level.

*we learn by
? doing*

VIII. *Focus.*—Attention should be focused on large purposes to be accomplished rather than on details. It would be splendid if we could analyze our golf game to the details and master these first, but unfortunately the human mind doesn't work that way; the beginner wants to "hit it a mile."

IX. *Difficulty.*—Interest instead of being opposed to effort is probably promoted by tasks which challenge the very best effort that pupils can put forth. If power is to be achieved, the hill must be long and fairly steep.

X. *Success.*—Power to handle a subject produces in the long run an interest in it. Associate success with the activity. Break up the activities into small enough tasks so that even low pupils can tell whether they are making progress. Use competition of groups, games and contests.

XI. *Participation.*—Interest is promoted by the pupil taking part in determining the purposes, in making the plans, in choosing the methods and appraising the results. James says "There is no impression without expression." Interest is the determining factor of discipline.

XII. *Dramatic Surprise.*—Interest is stimulated if recitations are conducted in such a way as to utilize the element of dramatic surprise. We must give the unexpected a chance to operate. There should be an informality of procedure, cheerfulness and sociability in the work. There should be much vivid illustration, humor, history and graphic material.

XIII. *Representation.*—Pupils like to stand for an idea or to represent some character that appeals. A boy who has taken the part of Thomas Jefferson in an imitation convention will probably talk and walk different for weeks. It may even affect his manners at the dinner table. A boy who gets himself associated with a "rough neck" trait is also under the control of an idea.

XIV. *Subject Matter.*—The teacher who has a fine control of

subject matter has great freedom to bring back to the fold the pupils with wandering attention. Moreover, it insures an extra challenge to the small group in each class possessing a deep permanent interest in the particular field.

XV. *Definiteness.*—*Everything else being equal, interest is promoted by definiteness of objectives.* Pupils like to be let in on the secret. A course should not consist of a series of "snipe-hunting" excursions.

(7) *The Report of the National Committee on Mathematical Requirements* has greatly promoted the mathematics movement. This committee was unique in a number of ways: First of all, it was the first committee dealing with a special high school subject to be adequately financed; second, it represented a fine cooperation between high school and college people; third, it set up two offices and obtained the services of well-trained workers for a period of years; fourth, the committee mobilized the forces emerging from approximately a hundred organizations interested in secondary school mathematics; fifth, the committee familiarized itself with the literature dealing with investigations and experiments; sixth, through preliminary reports the details in the final report were subjected to wide discussions and criticisms.

Now it is not to be inferred here that the Mathematics Committee said to the school men "Here is a course in general mathematics. Take it and teach no other." Far from it; but this group, consisting largely of conservative mathematicians, does include the following paragraphs in its report^a under the heading of "General" Courses:

"The movement has gained considerable new impetus by the growth of the junior high school, and there can be little question that the results already achieved by those who are experimenting with the new methods of organization warrant the abandonment of the extreme 'water-tight compartment' method of presentation.

"The newer method of organization enables the pupil to gain a broad view of the whole field of elementary mathematics early in his school course. In view of the very large number of pupils who drop out of school at the end of the eighth or the ninth school year or who

^a *The Reorganization of Mathematics in Secondary Education*—A Report by The National Committee on Mathematical Requirement, page 13.

for other reasons then cease their study of mathematics, this fact offers a weighty advantage over the older type of organization under which the pupils studied algebra alone during the ninth school year, to the complete exclusion of all contact with geometry."

(8) *The Trend toward General Courses in Other School Subjects.*—The growth of general science in our schools is well recognized. It would seem that arguments advanced for general science are equally good when applied to general mathematics. Even in the field of languages we hear discussion of "general" languages. At first hearing the phrase "general" languages probably sounds like nonsense to the conservative mind, but the writer knows a certain school in which very well-trained teachers in three different departments (English, French and Latin) adopted, by individual action and without conference, a general language book as a supplementary text for each group. The recent trend toward composite courses in the early years of college would seem to lend support to the position that general or basic courses have a place at the junior high school level. Hence we may reasonably expect at least a steady growth.

(9) *The Attitude of Former Critics.*—The fact that books, under the title "General Mathematics," and books for the seventh and eighth grades featuring general mathematics are being written by conservatives who in the early years opposed vigorously the movement tending toward correlation and general mathematics is perhaps the best evidence that the heart of the movement is essentially sound. It isn't every movement in education that makes vigorous enthusiasts out of its bitter critics.

Has There Been a Change with Respect to the Place of General Mathematics?—It has already been suggested that a change has taken place because of the coming of the junior high school. Ten years ago, not many high schools controlled the seventh and eighth grades. But such basic work is now given in many schools in the seventh and eighth grades. We have in this shift a similar illustration in general science originally designed for the ninth grade but also shifting in many schools to the seventh and eighth grades as soon as these grades were available. As far as the writer can see, there is no distinction between junior high school mathematics and the material which seventh and eighth grade pupils should study if the same pupils were housed in an ordinary

elementary school building. In other words, assuming a given number of children in a particular grade, children in a certain community and with a special teacher, why should they study one type of mathematics if housed in a grade building and an entirely different kind if reciting in a room in the junior high school?

In the ninth year, the trend seems to be to assign the students of lower intelligence levels to general mathematics. There is now very little difference between the third year books of some junior high school series and the formal algebras that are written for conservative schools. The general mathematics movement, along with other forces, has driven conservative writers to accept important modifications. For example, no algebra has been published during the last two years which does not include about a month's work in numerical trigonometry and some also include numerous geometric illustrations and graphs. There certainly is a large group of pupils in the ninth year who cannot profitably undertake a formal course in algebra. It may be that their needs will be met by further modifications in the old-time algebras or by definitely enrolling them in easier general mathematics courses. The hope of meeting their needs in commercial mathematics courses seems at this time vain for the very good reason that no one appears to know what commercial mathematics is and to no considerable extent have commercial arithmetics recently made raids on materials commonly classified as general mathematics. Possibly the smaller group of students with special interests in mathematics (those who will be teachers of mathematics, students of the advanced sciences, and the like) may profitably devote their time to the formal type of algebra. But the experience of teachers of correlated, fused or general mathematics seems to be that even these pupils can get a more intelligent basis for later work by an emphasis on the basis and material phases in the early years.

By way of summary, general mathematics has a place in the seventh and eighth grades (in the junior high school and in the conventional elementary schools if teachers are able to teach it) and it has a place in the ninth grade for the large percent of the students that have no special interest or ability in mathematics.

What About Adjustment with Senior High School Courses?—First of all, teachers in the senior high school need to realize that

both the National Committee, representing the American Mathematical Association, and the College Entrance Examination Board have broken away from excessive formalism. If senior high school people reorganized the second course of algebra as urged by the National Committee on Mathematical Requirements, less skill and more understanding would be required of pupils coming from the junior high school. The lack of adjustment between the junior and senior schools is often due to the fixed habits which dominate teachers. For example, there are many teachers in the senior high school who discuss the new programs advocated by the college men with considerable enthusiasm, but then go back to their classrooms, the slaves of habit, and teach the same extensive lists of cases in factoring, the complex fractions and the "nests" of parentheses. These still expect pupils coming from junior high schools to know these many complex factoring cases in spite of the fact that college authorities advocate only three simple types. Hence we often hear the opinion expressed by senior high school teachers, "the pupils coming from junior high school do not know anything." In part the solution may lie in widespread discussion. The organization of city mathematics clubs, in which the teachers of both junior and senior high schools participate, seems to have ironed out the differences in many cities.

It is granted that in schools where geometry is taught in the tenth grade pupils coming from the junior high school show considerable advantage from their extensive training in the intuitive geometry taught in the junior high school. But to the writer the extent of this knowledge, if it exists, has not been measured.

It has already been pointed out that books representing revisions of traditional algebra texts widely used differ but little from the third books in several series written for the junior high school.

There are, however, schools which offer two courses for ninth grade pupils: (1) an easy general mathematics course, and (2) a course in algebra as modified by the recent requirements. In a case where a pupil has completed one year of general mathematics and wished to continue his study of mathematics (perhaps he has changed his mind and now wishes to go to college), it is fairly common practice to enroll such a pupil in the second half year of ninth grade algebra. He loses no credit toward gradua-

tion but of course must obtain the required number of specific units for college entrances.

1) *What About College Entrance?*—There are a considerable number of sets of books written for the junior high school which contain enough algebra distributed throughout grades seven, eight and nine so that a principal can unhesitatingly certify this work as the equivalent of the unit of algebra as defined (1) by the College Entrance Examination Board and (2) by the National Committee on Mathematical Requirements.

A caution may not be out of place at this point. Test results, as well as common sense, tell us that pupils forget mathematics and all other things very rapidly. If a pupil takes a course in general mathematics (or algebra in the ninth grade) and no mathematics in all the years that intervene between this and the time of college entrance, he must not be expected to know much mathematics. A three-year span in any subject will cause pupils to forget most of the things imperfectly learned in the ninth year. Pupils going to college are in some schools advised to take a second course in algebra and to take that course just as close to college entrance as possible.

Some years ago when junior high school mathematics was first discussed there were some professors of mathematics of college courses who were bitterly critical. They gave as evidence against junior high school mathematics the very low mastery exhibited by college freshmen coming from the high schools. These critics of the junior high school overlooked the all-important fact that the product of high schools was the outcome of "good old-fashioned materials and methods." The whole reform was directed at the very thing of which they were complaining and which they were using as a means of suppressing general mathematics and the junior high school. Some years ago a head of a mathematics department in a great university stated in a large section meeting that general mathematics was the direct cause of low scholarship in college mathematics, but later was forced to admit that only 2 percent of his freshmen had at any time been enrolled in general mathematics courses. His students, nearly all of them, had studied traditional arithmetic in grades seven and eight, algebra in nine and ten and generally in grade eleven. But he was thoroughly discouraged with the results and well might be, for he was reviewing conditions which Professor Moore recognized twenty-four years ago when he made his

brilliant suggestions for remedial measure and which to-day are rapidly being achieved in some respects through the new courses in general mathematics.

The mistaken notion that it is necessary to formalize the teaching of mathematics in the junior high school grades in order to make a good showing on College Entrance Examinations needs to be overthrown. There is now substantial evidence in the reports from schools which are giving unconventional courses in these grades to show conclusively that those pupils who took these courses in the junior high school made good records later with reference to the College Entrance Examinations. Consider first the English High School (Boston). It has long had a ninth grade course with considerable innovations but those graduates do creditable work at the Massachusetts Institute of Technology. The graduates of the University High School of Chicago seem to have an excellent record at the University of Chicago. These graduates have been trained in a course featuring correlation for four years and have not taken algebra or geometry as such at any time. On the other hand, the standing of the Mathematics Department of the University of Chicago leaves no doubt as to the type of preparation required. The Lincoln School (New York City) has taught general mathematics in grades seven and eight and a modern type of algebra in grade nine since 1917. The graduates of this school have a most excellent record in mathematics in Harvard, Yale and other eastern institutions and especially on College Entrance Board Examinations. No special instruction has ever been given, nor has the school permitted the usual form of tutoring for such examinations. The graduates of this school not only pass the College Entrance Examinations but are given advanced credit at colleges of the first rank. Consider, finally, the Cass Technical High School (Detroit). It has for more than ten years operated with a mathematics course in ninth grade very different from the formal algebra. Many of its pupils come to the University of Michigan and Cass Technical High School has an excellent record at the University of Michigan.

It is possible that the indictment that college entrance examinations are not the stimulus toward valuable work in mathematics, which they might well be, can be substantiated. But bad as they may be, they are not the obstacles to worth-while courses in mathematics that they are frequently believed to be.

RIGOR VERSUS EXPEDIENCY IN THE PROOF OF LOCUS ORIGINALS

By ELMER R. BOWKER,

Public Latin School, Boston, Mass.

A modern textbook tells us that geometry is the study of the most efficient methods of dealing with the shape, size, and position of objects. I have no fault to find with this definition for a beginner of the subject. But from the point of view of the philosopher the definition is too inclusive. It implies that there is but one geometry when, as a matter of fact, there are several, for example, the geometries of Euclid, Riemann, Lobatchewsky, and Bolyai. The difference in these geometries lies entirely in the assumptions on which they are based. The method of each is the same in that a series of propositions is proved by deductive reasoning from the original premises. We teach a modification of Euclid's geometry in our schools because the results obtained by reasoning from his premises are more in accord with the evidence of our senses and experience.

For the moment let us say that any geometry is a series of propositions arranged in logical sequence and built up by pure reason upon a basis of definitions, general axioms, and specific postulates. Now if the argument developed is to be free of contradictions and to rest upon a minimum of assumption, the postulates must be few in number and carefully tested for consistence and redundancy. To be consistent, no postulates may lead to logical contradictions. To be free from redundancy, no postulate may be derived from another; it must be entirely independent or else proved as a proposition.

No body of consistent theorems can be developed from postulates which lead to contradictions. Redundant postulates on the contrary may be used without harm to the logical development or to the results obtained.

In building a geometry or in studying an existing geometry we have thus two methods of attack. If we desire an entirely rigorous development, we shall start with postulates which are not redundant and prove as propositions those theorems which

might be immediately inferred from these postulates. Such a procedure necessitates the proof of obvious theorems such as, "Only one perpendicular may be erected at a given point in a line," or, "The sum of any two sides of a triangle is greater than the third side." This method may be used with profit by mature students but it is entirely unfitted for beginners in our high schools. Nothing is more deadly to the interest and curiosity of our fourteen-year-old pupil than the futile and tedious proof of what to him is obvious. Time is wasted. Confusion results in attempting to prove by hair-splitting logic what the pupil would gladly accept without question.

The other course open to us is to accept without proof what is easily understood and proceed immediately to the proof of theorems whose truth is not readily seen. We postulate freely and worry not at all if we have redundancies. What of it? Our pupils are using the method of geometry. They are thinking geometry. Their interest in the new subject is kept alive by sensing its power.

Happily we have in our schools adopted this second method of attack. Most of our books have redundancies in their assumptions. But we are not greatly concerned. We abandon the rigorous method for the more teachable and expedient one—for the good and sufficient reason that good pedagogy demands it!

Why, then, have we retained the rigorous method in the proof of locus originals when the history of locus teaching, in our schools, is a record of consistent failure! We have not even succeeded in teaching the plotting of a simple locus let alone the more difficult business of proving it after it is plotted. If any evidence is needed to support this assertion, it can be readily obtained. On the 1926 Mathematics C—Plane Geometry paper of the College Entrance Examination Board question number 5 read as follows:

Find the locus of points from which the tangents to a given circle of radius 5 inches are 8 inches long. Give proof.

In the latter part of November, 1926, the College Board sent out a circular containing eight pages of "Solutions" of this problem offered by the candidates. There are eighty-one different attempts to plot this locus and only one of them comes anywhere near the solution. These candidates probably knew that locus was a word of five letters. Beyond that they had no clues.

Collect the aimless drawings of eighty-one four-year-old children and you have as good a set of solutions as these candidates offered.

It might be said here that the time we spend in trying to teach the proof of locus problems could much better be spent in teaching our pupils to plot a locus successfully. The concept of locus which we are agreed is essential to a course in geometry can be gained by learning to plot a locus. The business of proving it adds little. If we wish to test our pupils on their ability to build up a synthetic proof, let us test them with originals based on the regular propositions. Note, however, that the problem quoted above ends with the fatal words, "Give proof." So prove it we must!

Now when the 1926 group of geometry readers for the Board were fixing criteria for the marking of the proof of this question, they found two distinct types of proof to deal with: one, which I shall hereafter call the rigorous method, was based upon a statement of the dual conditions for establishing a locus with the resultant attempt to prove these two; the other, which I shall call the expedient method, was based upon the quotation of this postulate as authority, "The locus of a point at a given distance from a given point is a circle whose center is the given point and whose radius is the given distance." Candidates received all the credit on the question (a) if they stated the dual conditions and proved both of them and (b) if they based their proof on the "circle as a locus." Candidates who stated the dual conditions and proved only one of them received most of the credit. By this tacit admission of the inability of most pupils to prove both conditions this group of readers establishes one of the premises of my argument for a general adoption of the expedient method which I shall propose. It would be interesting to have a count made of the number of candidates who were able to state the dual conditions and were then unable to prove one or neither of them. I believe it would back up this assertion—that fifty percent of the pupils who state the dual conditions are unable to prove them. Add to this the number who are unable to plot the locus at all and the percent would probably rise to seventy-five. And yet we proceed merrily on our way requiring and attempting to teach the rigorous proof. Is this good pedagogy?

For purposes of comparison and contrast permit me to make a brief exposition of the rigorous method of proof for loci. Please keep in mind throughout the remainder of this discussion that the plan I propose has to do with the proof of locus originals only, not propositions. For the proof of locus propositions we must still resort to the dual proof. I do not wish to consign the rigor of geometry to the eternal void. I am not arguing for a geometry of mush. But there is nothing to gain in attempting to prove the unprovable or in attempting to teach the unteachable.

THE RIGOROUS METHOD

In outlining the proof for a locus theorem most textbooks assert that two conditions must be proven to establish the locus completely. One writer calls one of these conditions "sufficient" and the other "necessary." As a matter of fact two are necessary, so neither one can be sufficient.

The Wentworth-Smith book, so long a standard in our schools, cites the following: To establish a locus we must prove

1. (Theorem) *That any point in the supposed locus satisfies the condition.*
2. (Opposite) *That no point outside the supposed locus satisfies the given condition.*

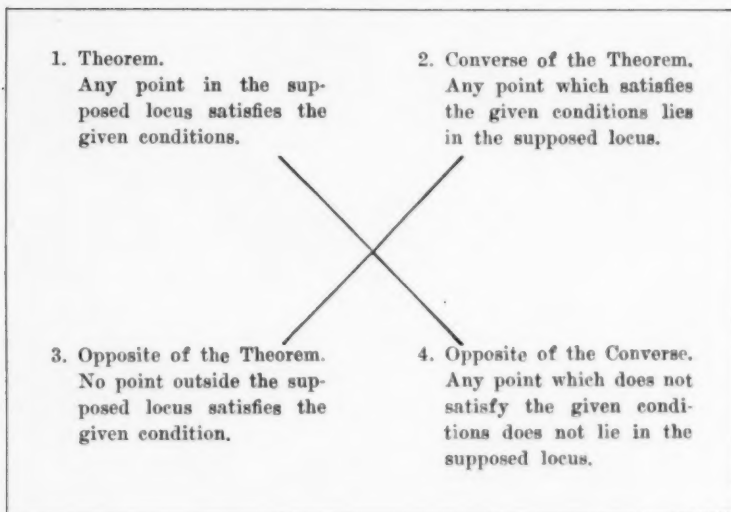
The Durell-Arnold book, one of the best modern texts, says:

It is necessary to prove

1. (Theorem) *That every point in the given line satisfies the given law or condition.*
2. (Converse) *That every point which satisfies the given condition lies in the given line.*

They do not go far enough. The possible thirty percent of any class who will be able to apply this method to locus originals would appreciate and should be shown the following:

Every theorem in geometry is related to three others in the following way.



The diagonal lines indicate that

If 1 is true, then 4 is true.

If 2 is true, then 3 is true.

Obviously then if we wish to establish the truth of the group, we may do it by proving any two adjacent ones as we go around the square; such pairs as (1, 2), (2, 4), (4, 3) or (3, 1). For the proof of some originals it is much easier to use the pair (1, 2) and for others it is easier to use the pair (1, 3). If we desire rigor plus efficiency for that upper third of our classes, these pairs should certainly be shown.

THE EXPEDIENT METHOD

The method of proof for locus originals which I propose is briefly as follows:

1. Insert in the syllabus in the proper logical sequence seven fundamental locus theorems.
2. Prove locus originals by quoting these theorems.
3. Require no original locus proofs which cannot be handled by quoting these theorems.

The seven fundamental locus theorems which I propose were first stated in a Report of a Committee on Locus Problems pub-

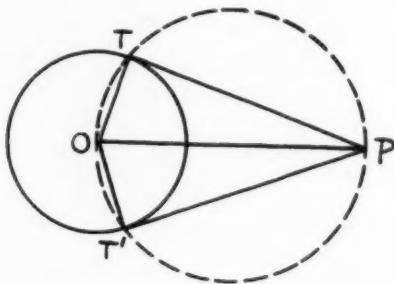
lished by the Association of Mathematical Teachers in New England. This pamphlet was distributed to the members of the Association and thus came to be used in a few schools. It has not had, I believe, a wide circulation outside of New England. The Theorems follow:

1. The locus of points at a given distance from a given point is a circle whose center is the given point and whose radius is the given distance.
2. The locus of points at a given distance from a given line is a pair of lines, one on either side of the given line, parallel to it, and at the given distance from it.
3. The locus of points equidistant from two given points is the perpendicular bisector of the line joining these points.
4. The locus of points equidistant from two given intersecting lines is the pair of lines which bisect the angles formed by the given lines.
5. The locus of points equidistant from two given parallel lines is a line parallel to the given lines midway between them.
6. The locus of the vertex of a right angle whose sides pass through two given points is the circle whose diameter is the line joining these two points.
7. The locus of the vertex of an angle of given magnitude whose sides pass through two given points is a pair of equal arcs passing through the two points and on opposite sides of the line joining them.

Numbers 3 and 4 are already proven in book one by the dual method. Number 1 is a book two postulate. Numbers 6 and 7 may easily be inserted as corollaries to the angle measurement propositions. Number 7 is no longer required by the College Board but it is an exceedingly useful theorem for construction work. We must not forget that we are teaching geometry, not an examination subject. The others, numbers 2 and 5, are fundamental notions about locus which every good teacher develops in the course of his teaching. Include all of them, however, either as postulates, corollaries or propositions. Prove them if necessary but—include them. Give them the status of established authority which may be quoted in later argument.

With these seven theorems concerning locus as established authority, the pupil is now equipped to prove locus originals with more hope of success. Let us see how the new method works. In the exercises which follow, the description of the locus and its plotting is omitted since it is the proof that we are interested in. I do not offer these as model proofs, but merely as illustrations of the proposed method. Doubtless many teachers would be able to suggest improvements in arrangement.

1. Find the locus of all points from which tangents of a given length can be drawn to a given circle.



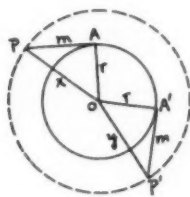
Given: Circle O with radius r ; tangents of length M are drawn to the circle O .

Required: The locus of the point P , as the tangent M moves around the circle.

1. Take any two positions of the tangents such as AP and $A'P$.
2. In triangles OAP and $OA'P$, r is perpendicular to m .
3. $y^2 = r^2 + m^2$ and $x^2 = r^2 + m^2$.
4. $y^2 = x^2$.
5. $y = x$.
6. The locus of the point P is the circle with O as a center and a radius x .
2. The tangent to a circle at a given point is perpendicular to the radius drawn to that point.
3. In a right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.
4. Things which are equal to the same thing are equal to each other.
5. Like roots of equals are equal.
6. The locus of points at a given distance from a fixed point is a circle with the fixed point as center and the given distance as radius.

Q. E. F.

2. Find the locus of points of contact of tangents drawn from a given point to concentric circles.



Given: Concentric circles O and tangents drawn from the external point P .

Required: The locus of the points of tangency T .

- | | |
|--|---|
| 1. The line OT always passes through O . | 1. OT is a radius. |
| 2. The line PT always passes through P . | 2. Given condition—tangents are drawn from P . |
| 3. In any position of PT , angle OTP is a right angle. | 3. The tangent to a circle at a given point is perpendicular to the radius drawn to that point. |
| 4. The locus of T is a circle on OP as a diameter. | 4. The locus of the vertex of a right angle whose sides pass through two given points is a circle on the line joining the points as diameter. |

Q. E. F.

These two problems are sufficient to illustrate the working of the method. I have been using this method of proof in my classes for four years and I know of other teachers who use it. My experience has been that the pupils are much more successful in its use than they ever are with the dual proof.

It may be argued that the number of locus originals available for proof will be greatly reduced by restricting the field to those originals which can be proved by quoting these seven theorems. This may be true. I believe, nevertheless, that there are plenty of originals to be proved by this method. Most of our text books exhibit a lack of locus material anyway. Any teacher who expects to obtain results from his locus teaching is obliged to search for more material. Let him search further. I repeat my previous assertion that the ideas we gain from the study of locus are gained from the plotting and not from the proof.

My proposal does not call for the discarding of the dual proof for propositions. But for locus originals let us abandon it. The

majority of the pupils are unable to use it anyway. It is the old argument of rigor versus expediency. We do not begin our courses in geometry with a rigorous investigation of the foundations. We assume a great many things because that is the teachable method. Is it a softer psychology to reduce the difficulty of the locus proof? Give the pupil and the teacher a fighting chance to succeed!

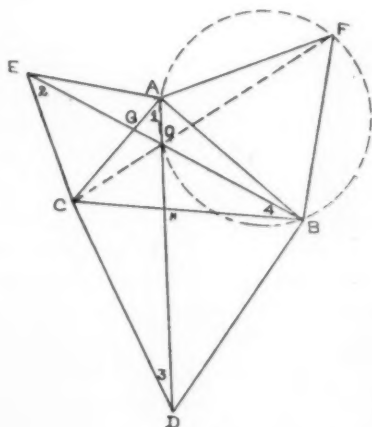
PROOF OF AN ORIGINAL EXERCISE ¹

BY MASTER WALTER BEYER,

Milton Academy

Hypothesis: On the sides of any $\triangle ABC$, equilateral $\triangle AEC$, CDB , BFA are constructed.

To Prove: AD , BE , FC are concurrent.



Proof.— Statements

Let AD and BE intersect at O .

Draw OF and OC .

$\triangle AEC$ and CDB are equilateral.

In $\triangle ECB$ and ACD ,

$AC = EC$, $DC = CB$.

$\angle ECA = 60^\circ$,

$\angle BCD = 60^\circ$;

$\therefore \angle ECA = \angle BCD$,

$\angle ACB = \angle ACB$;

$\therefore \angle ECB = \angle ACD$;

$\therefore \triangle ECB \cong \triangle ACD$;

$\therefore \angle 1 = \angle 2$, $\angle 3 = \angle 4$.

In $\triangle EGC$ and AGO ,

$\angle 1 = \angle 2$,

$\angle EGC = \angle AGO$;

$\therefore \angle AOG = \angle AEO$.

$\angle EAC = 60^\circ$;

$\therefore \angle AOE = \angle EAC$,

Reasons

Hyp.

Sides of equilateral \triangle are $=$.

Each \angle of equilateral $\triangle = 60^\circ$.

Same reason.

Axiom.

Identity.

Add. Axiom.

s. a. s. = s. a. s.

Corresp. parts of $\cong \triangle$ are $=$.

But $\angle ECA = 60^\circ$;

$\therefore \angle AOG = 60^\circ$.

In $\triangle AEO$ and AEG , $\angle AOE = 60^\circ$,

Just proved.

¹ Proof of the "original" announced in the December 1926 issue of THE MATHEMATICS TEACHER.

$$\begin{aligned}\angle AEO &= \angle AEO; \\ \therefore \triangle AEO &\sim \triangle AEG; \\ \therefore EO:EA &= EA:EG.\end{aligned}$$

$$\begin{aligned}\text{But } EA &= EC; \\ \therefore EO:EC &= EC:EG, \\ \angle 2 &= \angle 2; \\ \therefore \triangle EOC &\sim \triangle EGC;\end{aligned}$$

$$\begin{aligned}\therefore \angle ECG &= \angle EOC. \\ \text{Since } \angle ECG &= 60^\circ, \\ \angle EOC &= 60^\circ, \\ \angle HOB &= \angle EOA, \\ \angle EOA &= 60^\circ, \\ \angle HOB &= 60^\circ.\end{aligned}$$

But $\angle AOB$ is supplementary to $\angle HOB$.

$$\begin{aligned}\therefore \angle AOB &= 120^\circ. \\ \triangle AFB &\text{ is equilateral.} \\ \therefore \angle AFB &= 60^\circ.\end{aligned}$$

$\angle AFB$ is supplementary to $\angle AOB$.

$$\begin{aligned}\text{In the quadrilateral } AOB F, \\ \angle AFB + \angle AOB + \angle OAF \\ + \angle OBF &= 360^\circ, \\ \angle AFB + \angle AOB &= 180^\circ; \\ \therefore \angle OAF + \angle OBF &= 180^\circ, \\ \text{i.e., } \angle OAF &\text{ is supp. to } \angle OBF; \\ \therefore \text{ a circle can be circumscribed} \\ &\text{ about } AOB F.\end{aligned}$$

Then $\angle ABF$ is measured by $\frac{1}{2}\widehat{AF}$.

$$\begin{aligned}\angle AOF &= \frac{1}{2}\widehat{AF}; \\ \therefore \angle ABF &= \angle AOF. \\ \text{But } \angle ABF &= 60^\circ; \\ \therefore \angle AOF &= 60^\circ. \\ \text{Since } \angle EOC &= 60^\circ, \\ \text{and } \angle AOE &= 60^\circ; \\ \therefore \angle COF &= 180^\circ; \\ \therefore COF &\text{ is a straight line passing} \\ &\text{ through } O; \\ \therefore AD, BE, FC &\text{ intersect at } O, \\ \text{i.e., they are concurrent.}\end{aligned}$$

Vertical \angle are $=$.

If two \angle of one \triangle = two \angle of another, the third \angle are $=$.

An \angle of equilateral \triangle = 60° .

Substitution.

Proved above.

An \angle of equilateral \triangle = 60° .

Sub.

Identity.

Two \triangle are similar if two \angle of one = two \angle of the other.

Corresponding sides of similar \triangle are proportional.

Sides of equilateral \triangle are $=$.

Sub.

Identity.

Two \triangle are similar if an \angle of one = an \angle of other, and including sides are proportional.

Corresp. \angle of $\sim \triangle$ are $=$.

An \angle of equilateral \triangle = 60° .

Sub.

Vertical \angle are $=$.

Proved above.

Axiom.

Two adjacent \angle with their exterior sides in st. line are supp.

120° is supp. to 60° .

Hyp.

An \angle of equilateral \triangle = 60° .

$60^\circ + 120^\circ = 180^\circ$.

Sum of \angle of quadrilateral = 360° .

Proved above.

Subtraction Axiom.

If the opposite \angle of a quadrilateral are supplementary, a circle can be circumscribed about it.

An inscribed \angle is measured by $\frac{1}{2}$ its intercepted arc.

Same reason.

Axiom.

Sub.

Proved above.

Proved above.

The whole = sum of its parts.

Q. E. D.

THE ARITHMETICAL PRODUCTIVENESS OF UTIL-
ITARIAN, SOCIAL AND SCIENTIFIC IDEALS;
VIEWED HISTORICALLY

By PROFESSOR G. W. MYERS,

The University of Chicago

The Beginnings.—The history of arithmetic, as such, really begins with the devices and technics that were created, collated and employed by the Phœnicians. The demands of their commercial transactions between Sidon, Tyre, Carthage, and other Mediterranean cities were extensive and made considerable demands for numbers. The affairs of this ancient sea-faring and commercial people called for account-keeping, record-making, money-exchanging and for figuring balances. This demanded a number symbolism and a system of weights and measures. Out of these needs developed the beginnings of an arithmetic. The Phœnicians had no body of calculatory rules nor any laws applicable to number relations. They were in fact not conscious of more than a loose collection of procedures to be acquired and transmitted through a trade apprenticeship. They had not even given any distinctive name to their body of empirical rules. They felt no other than a utilitarian need and required only a conventional number symbolism and a crude abacus. We are told that the Tyrians regularly supplied the weights and measures used in Babylon. The utilitarian ideal here produced only the most meager beginnings of calculatory procedure. It was not strongly creative.

The Babylonians.—The ancient Babylonians were of a remarkably practical turn of mind. They were predominantly merchants and agriculturalists. They were also strongly religious. Their recognized rulers had to be priests first, deriving their kingly right from the divine source. Personifying their gods as the sun, moon and other heavenly objects, as they did, their attentive interest and concern were drawn to the heavens. Thus they became the first astronomers. Since their astronomical interest was mainly religious, they were only observing and recording astronomers. Their religion and astronomy as well as the documents needed in their mercantile and agricultural activities, all called for permanent records and a chronology. Merely to supply these needs they formulated a number system and a symbolism. They became most expert of all ancient peoples in mercantile calculations. They developed no interest in their

calculatory technique itself. They developed no arithmetic, and always employed calculations as a mere subsidiary art. The utilitarian and social ideals here produced nothing beyond what would meet the immediate need. As actuating mathematical ideals they were nearly futile.

The Egyptians.—The ancient Egyptians were an agricultural people. For centuries the rich alluvial of the Nile valley made Egypt the granary of the civilized world. The people were mainly interested in the areas of fields and the contents of barns and granaries. They had their professional "rope-stretchers" to function as official surveyors and gaugers. They had partly borrowed and partly devised a sort of clumsy symbolism and calculatory procedure to perform these simple offices.

The Egyptians were also worshippers of the heavenly bodies and students of the heavens. These interests involved the orientation and erection of religious temples and monuments. Their orientations were with reference to the solstices, the equinoxes or the pole-star. They developed or borrowed enough simple geometry to enable them to do these things, but they evinced no concern to digest their rules into any sort of order, nor did they formulate any astronomical system. Simple calculatory procedures withal very crude sufficed their need, together with a very little practical geometry, much of which was erroneous though under actual conditions roughly correct. The utilitarian ideal with them did not go beyond their immediate needs. The Egyptian felt no impulse to develop any basic science under his crude calculatory art. He furnished the world its most marvellous example of an amazing proficiency in art based upon no scientific groundwork. The Egyptian made no progress in mathematics beyond the most meager beginnings. The utilitarian ideal here again proved an inadequate mathematical motive.

The Early Outcome.—The ancient Phœnicians, Babylonians and Egyptians thus furnish us the earliest and the most outstanding examples of the truth that purely practical standards of culture are very feeble growth factors for mathematics. Only the feeblest calculatory beginnings, such as trade and barter made imperative, issued from these standards. Because these peoples did not think through their practical art down to its scientific meanings and bearings, they were really *artisans* rather than *artists*. Their scanty culture became stagnant at a very

primitive stage. Mathematical culture does not break through rote practices into successive stages of growth without the aid of many vigorous acts of reflective thought at each level that marks the stage. The impulse to introspective analytic thought was lacking amongst these early peoples, social and utilitarian standards dominated them, and mathematics could not germinate and grow for lack of a suitable incentive. All that can be expected to come from social and utilitarian ideals is systems of record, metrical technics and calculatory procedures, which are the art phases of arithmetic only.

Theorizing Begins.—Thales, the first great Greek scholar, was primarily a philosopher and secondarily an astronomer and mathematician. He was of Phœnician nativity, had been trained in the great schools of Asia Minor, and had the strongest native predilection for speculative thinking. After a successful business career, he opened a school of philosophy and devoted his life to philosophical pursuits. He found the few mathematical rules and facts of the Egyptians the best raw material with which to begin his philosophical studies, and on which to base his teachings. He probably created the word "mathematics" to connote what we mean to-day by "general science." Through him and his school of philosophy the word gained currency in this connotation amongst the Greeks, for his school was in Greece. Thales and his disciples gave considerable study to the theory of the calculatory practices of the Phœnicians. No important advances in arithmetic however are on record as having come from these theoretical studies until they appeared in the work of Anaxagoras on squaring the circle and in the performances of the Pythagorean school.

Art Begets Science.—Pythagoras, a pupil of Thales, was the next great contributor to this new and forming science of mathematics. Pythagoras' father was a Phœnician, though his mother was a Greek. The outstanding characteristic of the ancient Greek mind was its speculative and scientific bent. Perhaps the law of inheritance of parental traits, that sons derive their distinctive traits from the mother, was operative in the case of Pythagoras. History has it that Pythagoras was well versed in the teachings of the great schools of western Asia Minor. He knew their calculatory number technics and was familiar with the doctrines of the school of Thales, when he took up his life pursuit of philosophical study. He had also steeped himself in

the lore of Egypt and probably also of Babylon. His native bent to philosophic inquiry was turned upon the new mathematics. He divided the science into 4 branches, viz.: arithmetic, music, geometry and astronomy. These branches were what later became the "*Quadrivium*" of secondary cultural programs.

Arithmetic as used in Pythagoras' classification meant not calculation, but *theory of numbers*. The Greeks probably taught children calculation with the abacus, but Pythagoras did not consider it worthy of mature contemplation. He called calculation 'logistic,' and theory of numbers 'arithmetic,' and in philosophical schools they studied only arithmetic. Thales and Pythagoras were then the originators of theoretical arithmetic. The creative interest was the offspring of the scientific, not of the utilitarian or social ideal.

In accord with the law of historical genetics we have here an instance, rather notable, of the art begetting its related science, rather than the reverse, for this theoretical arithmetic had its origin in Phœnician calculatory technics and algorithms. These pure calculatory procedures in the mind of the speculative Greek grew into theoretical number considerations and later 'went to seed' in an unbridled mysticism.

As to Origins.—From what has been said it appears that in its historical genesis calculatory arithmetic (what the Greeks termed 'logistic') is Phœnician, not Greek, Egyptian nor Babylonian, as some writers seem to think. The Phœnician was driven to his inventions by the necessities of trade and his initiative ceased operating so soon as the necessities were met. He lacked the natural 'urge' to organize his possessions. In its theoretical character, however, arithmetic is Greek in origin. Greek scholars had a strong native bent for abstract speculation. Above all things they loved to theorize. They are the only nation of antiquity that made proofs a part of their scientific literature. They devoted themselves so exclusively and painstakingly to pure theorizing about the calculatory elements of arithmetic that their product was only a one-sided affair. These scholars so completely cut the theory away from its moorings in utility that to find application for the results they reached in their theorizing, they had to make mystical associations of number with personal qualities and philosophical notions.

Arithmetical Mysticism.—Pythagorean arithmetic was a peculiar mixture of sound reasoning and mysticism. The thinkers

of this school reasoned to sound conclusions about number and then made such assumptions as "eight is the cause of cold," "five is the cause of color," etc. They then sought to draw filmy inferences from such assumptions and were led to a spurious sort of philosophical mysticism. This arithmetical mysticism, which is the nearest mathematical analogue to the pseudo-sciences of alchemy and astrology that history affords, did not, however, long appeal to the Greek scientist and philosopher, nor did it ever subsequently commend itself to clear thinkers. It was practically discontinued with the close of the school that begot it. The after-effect is somewhat reflected in the fact that the next succeeding Greek schools turned away from it entirely and devoted themselves with some success to the study of the laws of pure calculation, to pure 'logistic.' This was a direct reaction from the excessive subjective theorizings of their forbears. Although the Sophists approved and taught almost everything else that the Pythagoreans left behind, they would have nothing to do with Pythagorean speculative arithmetic. Pythagorean mysticism accordingly furnishes us an outstanding example of two important truths, viz.:

1. Scientific advance can break through and go beyond a fixed stage only by means of a considerable amount of reflective and organizing thought upon the materials of human experience already at hand, and—

2. Cutting arithmetical thinking off too completely from utilitarian moorings and check-ups is scientifically disastrous. It is the utilitarian things that are of perennial concern to human beings from age to age and they must very largely determine the values for a given age of the contributions of a former period. They are the *anchorages* of theory to reality.

After the Sophists.—The Sophists made no great progress in their 'logistic.' Following the fortunes of arithmetic after the Sophists, we learn that Plato explicitly disapproved of the study of calculatory arithmetic in the words: "The study of logistic is unworthy a philosopher. It is a childish art." Eudoxus, a contemporary of Plato, does not seem to have regarded the study of arithmetic, including calculatory aspects, as "unworthy of a philosopher." He gave considerable attention to the study of irrational numbers, as we know from the arithmetical chapters of Euclid's *Elements* that were drawn from Eudoxus. Eratosthenes devised "The Sieve" for detecting primes, and Aristar-

thus made extensive use of calculation in astronomy, but he devised no new methods.

Archimedes made extensive use of calculation in his geometrical and mechanical studies, but gave no attention to calculatory processes in particular, excepting that his calculations of π seem to show that he had devised some peculiarly effective way of getting square roots. His *Sand Counter*, however, proposes a system of numeral notation which, in some respects, is much more powerful and lucid than was the current Alexandrian system. His uses of calculation were in the main of a non-mathematical character. To him calculation was a tool of no interest in itself. He cannot be said to have advanced arithmetic.

Apollonius wrote a small work on the methods of arithmetical calculation, which was merely a sort of ready-reckoner. In this booklet he proposed a system of numeral notation similar to that of Archimedes, and probably pointed out that a decimal system, involving only nine symbols, would simplify numerical multiplications. He did not, however, develop anything really new in arithmetic. He made considerable use of calculatory procedures in his researches in Conic Sections, but was interested in it only as a tool for geometry.

Finding no calculatory technique adapted to his work of cataloguing and charting star places, Hipparchus devised a process of calculating by half chords of double arcs, which was so useful for astronomy that Claudius Ptolemy easily converted it into a systematic method. Both these scholars, however, regarded the scheme merely as an artificial means of reducing observed star places, worthy neither of study for itself nor of being an acceptable topic of arithmetic. The Hindus, who were the born lovers of calculation and who devised our modern algorithms for conducting calculations, developed trigonometry out of the Ptolemaic system.

Hero was greatly interested in calculation as a subsidiary tool in his geodesic work, and he enriched calculatory arithmetic with a highly expeditious rule for calculating square roots, easily extensible to cube roots, and another for calculating the area of a triangle from its sides. Here again, we see an enrichment of calculatory arithmetic, just in the degree in which the enricher developed an interest in studying calculation for its own sake. Hero was stimulated by a strong practical need for the rules, and he pushed the development of his rules only far enough just to meet the need.

The First Arithmetic.—Nicomachus, a Jew, in 100 A.D. developed so strong an interest in arithmetic as to undertake to do for arithmetic something like what Euclid had done for geometry and Apollonius for the conics. He undertook to dissociate both calculatory and theoretic arithmetic from their subsidiary relations to other disciplines and to form a new science of arithmetic out of their union, and to give it a distinct treatment of its own. He did his work so well that since the appearance of his text arithmetic has been regarded as a suitable study for scholars. With Nicomachus the race become conscious of the existence of arithmetic as an independent mathematical discipline. Both calculation and theory play important rôles with Nicomachus, as they continued to do thenceforth.

Some Lessons for Teaching.—The history of arithmetic since the appearance of the first text is of course highly interesting and affords many a worthwhile suggestion for teaching. We have intended here only to trace the evolution of arithmetic in history with main reference to actuating ideals, down to the time when the race became aware that there was such a thing as an arithmetical discipline. The reader will, we think, be able to sense the following conclusions as being of some interest to teaching:

1. Prior to Greek influence the peoples concerned in the development of calculation were actuated entirely by the utilitarian and social standards. They developed only a loose body of calculatory rules, to only such an extent as the momentary need called for and such as would be picked up in an apprenticeship to a particular trade, or vocation. The rules were not organized nor systematized and calculators were satisfied with the tedious and clumsy ways of the abacus. The utilitarian and social ideals are not productive mathematically, so far as this period of arithmetical history goes!

2. When the destinies of arithmetic fell into the hands of the Greek, his speculative impulse dominated his interests to the extent of leading him to become entirely engrossed in number theory and logic, to spurn utility in his results as degrading to culture and not even to concern himself with checking up his logical deductions against objective facts. The result of this nearly complete ignoring of practical calculation as a suitable theme for philosophical study was the development of a lopsided theoretical product that went to seed in arithmetical mysticism. This pseudo-science scarcely outlasted the school that invented

it. Greek arithmetical history furnishes a clear illustration of the disaster that comes to arithmetic from cutting its study entirely loose from the calculatory aspect of practical application. For the health of arithmetical theory alone continual objective calculatory check-ups are necessary.

3. Not until calculation and theory were unified into an organic body of practical rules and theoretic principles, each aspect assisting and illuminating the other, did arithmetic attain to better than haphazard empirical progress. Race experience with the subject to the time of Nicomachus should lead the modern teacher of arithmetic to see that successful arithmetic is that which is "the theory of numbers and the art of calculation." In this day of the testing mania we need often to stress the conjunction suggested in the quotation.

Recapitulatory.—From the earliest times the use of the abacus made it possible to carry out the simple addings and subtractings of early mercantile transactions without any knowledge of theoretic arithmetic. Social and community approval were upon current ways of doing the needful things and this sort of approval was dominant in business affairs. There was neither need nor desire for inquiring into reasons for conducting numerical manipulations thus and so. Social and community ideals alone delay rather than accelerate arithmetical progress.

Not reason but social approval was the recognized authority until Greek times. The route to calculatory skill was not study but apprenticeship. Arithmetic therefore grew very slowly and only by accidental accretions and when under Pythagoras it did start to grow the growth was directed to the wrong goal. The calculatory art was too fully ignored, or if not ignored, it was brought to play upon subjective matters, and therefore theoretic deductions were not checked up by objective facts. Consequently growth did not proceed far. The mystical arithmetic of the Pythagoreans is the sole product of scientific activity in ancient Greek mathematics, that languished and died in the hands of subsequent scholars. History, even thus early, signals to us the danger of an extreme dissociation of the theory of arithmetic from its practical calculatory side, or from its application to the objective utilities of daily life.

Conclusion.—A successful arithmetical doctrine must be broadly enough conceived to comprise all three ideals, the social, the utilitarian and the scientific.

THE USE OF PROBLEMS IN TEACHING ELEMENTARY ALGEBRA *

BY PROFESSOR E. H. TAYLOR,

Eastern Illinois State Teachers College, Charleston, Ill.

Elementary algebra should begin and end with the solution of problems. That is my text.

Many a boy in a poorly taught country school has found his chief interest in arithmetic. Arithmetic was the only subject that presented problems. His history was only an answer book, a long series of results of problems that he had no hint of. The arithmetic problem had an intellectual or practical appeal. So he tugged away at it and got the most of his interest, enjoyment, and training from arithmetic, from solving problems in arithmetic.

Elementary algebra makes its chief appeals to American high school pupils through its use, and through "the love of thought for thought's sake by those who can play the game of thought well." Both appeals are to the desire to solve problems.

What is a problem? First of all it must not merely call for a ready-made response. What is the cost of 3 oranges at 5 cents each?, is not a problem for the persons in this room. What is the length of an ellipse of eccentricity 0.6 and major axis 10?, may be. So teaching pupils to solve problems means teaching them to think through new relations. This is sometimes forgotten and pupils are thought to be reasoning in mathematical problems when they are only following a type solution that has been memorized. When a pupil says since $\frac{4}{5}$ of a number = 20, then $\frac{1}{5}$ of the number = $\frac{1}{5}$ of 20, he is not thinking numerical relations, he is trying to repeat a form. He may even memorize the form, remember to divide by the number above the line, without doing any mathematical reasoning. Our attempts to teach problem solving result too often in merely teaching devices to save thinking. We want results and the pupil wants a method that works every time. So we teach him a device fol-

* Read at the Conference of Teachers of Mathematics at the University of Iowa, October 9, 1926.

lowed by a list of exercises in which that device works. There are no new relations so no real problem solving. The pupil is merely giving back *our* expressions of *our* thinking. But he gets the correct answer, which pleases us. He is saved the irksome experience of trying to think, so he is happy. That makes it unanimous.

A problem then requires getting the answers to new questions. A problem by this definition need not be mathematical. Most of us are solving problems every day, that is, getting new conclusions from old facts. Suppose a real estate agent tries to sell me a house and lot. If I have a ready-made response, yes or no, there is no problem. It may be that I am undecided, I have no satisfying response. Then I have a problem. What do I do to solve it? As I think about the question many ideas come up, and I select those that seem pertinent. That is, I get together the data,—original cost, my ability to pay, taxes, insurance, state of the market, probable rise and fall of the price in real estate, amount of rent I am paying, neighborhood, and so on. The facts that seem significant I attempt to give proper weight and I try to find useful relations among them. If from these relations I get a result that gives me a sense of relief, of mental satisfaction, I have a solution of my problem. I do not mean that the solution is necessarily the correct one. But it is a solution.

The method used here is the same as in solving an algebra problem.

"A ball is thrown upward with a velocity of 40 ft. a second. How long will it take for it to rise 50 ft.?"

What is the method of attack? First the facts must be brought together. Here are some useful facts: $S = v_0 t - \frac{1}{2}gt^2$, $g = 32$, $v_0 = 40$, t^2 means t times t . The pupil may not know these facts. Then he can not get the correct answer to his problem. He may not know how to use them when he gets them. In my real estate problem, I may not know how to use the facts about taxes and upkeep after I get them. Then I do not get the answer that I should. If the pupil does not know the meaning of the algebraic formula, the facts are useless. But if he has the facts and can find the proper relations between them, he arrives at a result that seems reasonable, and checks. His curiosity is satisfied. He has solved the problem.

I would not restrict the word problem to the ordinary "clothed

problem," infantile numerical relations swaddled in verbiage, such as this one: "A woman sold oranges and apples. The oranges were worth 5 cents a dozen more than the apples; and 7 dozen oranges were worth as much as 18 dozen apples. What was the price per dozen?" What better argument that algebra is of no use than that we resort to this type of problem to apply it, one that would never arise except by first setting down the answers and then wrapping them up in words? In this case there was an evident slip as the answers are the remarkable ones of $3\frac{2}{11}$ cents a dozen for the apples, and $8\frac{2}{11}$ cents for the oranges. This type of problem has only one excuse, it gives exercise in translating English into algebra. That is a very important exercise and to be emphasized. But it should and can be exercised in useful and rational ways. In this discussion of problem solving, I am not thinking about solving this type of problem except as it is offered openly and aboveboard as an interesting puzzle. I would offer some of them on that ground, and some pupils will find them interesting. But I should try to have them of more interest than the one I have given. That type has been greatly overemphasized and has helped to bring algebra into contempt.

I would give the word problem more content. Let us consider some of the uses of problems.

a. We solve algebra problems to get desired results, results that some one wants to use. The number of real problems that can be solved by the methods of elementary algebra and whose conditions are within the experience of ninth grade pupils is small. That is very evident to those who have tried to make problems and to those who have tried to find them in books. The application of formulas gives more genuine problems than any other field I know.

b. We use problems to arouse interest and to show a motive for learning new processes. I have used the golden section effectively in starting the study of quadratic equations. A rectangle is said to be most pleasing to the eye when its adjacent sides are the parts of a line that has been divided into extreme and mean ratio. This leads to a quadratic equation. That equation is written on the board and the class told that when in the course of the development any pupil wants to finish the solution of it he may. Some of the class will keep this equation in mind and

be ready to solve it before the general solution is finished. This device of starting with a problem is of course the one most frequently used for furnishing a motive for the study of mathematics.

c. Good inductive teaching is teaching through problems arranged in a series leading to a desired result and each of which the pupil can solve. This method is used in teaching meanings. Our attempts to teach the meanings of variation and function are good examples. Definitions here are quite useless until meanings are put into them by solving problems. Consider the formula for finding the horsepower of automobile engines. $H.P. = nd^2/2.5$, where n is the number of cylinders and d is the diameter of the cylinders in inches. The dependence of the horsepower on the number and size of the cylinders may be illustrated by such questions as these: Does the horsepower increase or decrease as n increases? As d increases? Will doubling the number of cylinders double the horsepower? Will doubling the diameter of the cylinders double the horsepower? Only by computing answers to such questions will the notion of the dependence of one variable on another, the notion of function, be made clear. Algebraic symbols are given meaning by use in thinking numerical relations. What number is 1 greater than a ? Which is greater, a^2 or a^3 ? If z is a positive integer, what is the next greater integer? I would cut out many of the verbal problems, the "clothed problems," and put more questions of this sort into elementary algebra. They are more interesting, they are more instructive, and they provoke more thought. One outcome of elementary algebra should be ability to think in general numerical symbols, to give general answers to questions. The kind of question just suggested gives practice in making generalizations.

d. There are some kinds of numerical relation that occur so frequently that pupils should learn to recognize them. If A can do a piece of work in 3 days and B can do it in 4 days, how long will it take them to do it working together? The relations involved here are in many problems, all those in which there is a combination of two or more forces to perform a task. "If you can not solve a problem, try to solve an easier one like it," a teacher once said to me. That is the best recipe I know. I will admit that it is very hard to apply sometimes. It is hard to make

the "simpler one like it." But I do think that our teaching can do something to make pupils conscious of this method. If they see that it is possible for them to arrive at the answers to questions that at first seem quite beyond them by making a set of easier questions, they may be encouraged to do some thinking of their own. In these work problems pupils usually add to get the answer because "working together" sounds like addition. The method of attack can be made clear by such questions as these: What part of the work can A do in 1 hour if he can do the whole work in 3 hours? If he can do it in 6 hours? In 25? $3\frac{2}{3}$? a ? $x + 3$? If we know the part that each does in 1 day, how can we find the part they both can do in 1 day? The complete general solution of that type of problem can be developed in a way that requires the pupils to do their own thinking and which makes the method of attack appear. The pupils are finding a general method of solving problems by actually solving problems of the type.

e. We are most familiar with the use of problems to fix processes, test their mastery, and to apply them. After teaching how to solve linear equations in two unknowns, we usually give a class a set of verbal problems to apply what has been taught. Just what is our object? If you were free to do exactly as you pleased, would you teach that set of problems? If the object is to improve skill in solving linear equations, could not that be done better by more drill on given equations? Or is it that these are problems that pupils will some time need to solve? What types of problems that anyone ever solves in practice can ninth grade pupils solve by using linear equations in two unknowns? If you think that they are easy to find, find or make a list of a dozen. There are problems in mixtures that some one conceivably might need to solve, such as: "How much milk testing 4 percent butter fat and cream testing 24 percent butter fat must be mixed to make 20 gallons that tests 20 percent butter fat?" We solve linear equations to find the intersection of two lines. We also solve them in determining constants, as in finding m and b if we know that $y = mx + b$, and that $y = 3$ when $x = 2$, and that $y = 6$ when $x = 1$. But we do not solve linear equations in two unknowns to find out how many girls and how many boys there are in a classroom of 42 pupils if there are 8 more girls than boys. In the actual situation we would

either know the answer before we asked the question, or we would count and find out. Now the topic of linear equations is easy and interesting as a bit of algebraic manipulation. There are enough genuine applications to show that the process may some time be useful. Is it not better to fix the technique of solving by solving given pairs and limiting the applications to problems that somebody actually may have to solve?

Thorndike * very properly raises the question as to whether all techniques should be followed by verbal problems. He takes as an example simultaneous quadratic equations. The older view was to formulate fantastic problems that had no reason for being except that they led to simultaneous quadratic equations. A newer view plots these equations and solves them to find the coordinates of the point of intersection. It certainly is preferable to solve problems of the latter type which are genuine applications. It seems sensible simply to omit applications to verbal problems of such techniques as have no genuine applications within the range of high school pupils.

It is generally agreed that if there is to be a transfer of training, if the training in one field is to be useful in another, the pupils should be made conscious of the elements that are to be transferred. The study of the transfer of training, of mental discipline, that is reported in the Report of the National Committee, makes it very clear that it is the opinion of the educational psychologists whose opinions are quoted that the question whether the training that we are giving in mathematics is to be useful outside of the mathematics classroom depends largely on the teaching. Mathematics may be taught so that the training carries over and again it may not. We must make very clear to the pupils the elements that we expect to transfer. If the method of thinking in solving problems is to be of use in thinking through other problems, the pupil must be made conscious of the method.

I have found these directions useful in teaching pupils how to solve problems.

1. Read the problem carefully.
2. Decide what is to be found.
3. Decide what facts needed in the solution are given in the problem.
4. Decide what other facts are needed, and find these facts.

* The references to Thorndike are to *The Psychology of Algebra* by E. L. Thorndike, New York, The Macmillan Co., 1923.

5. Determine the processes needed in the solution.
6. Estimate the result.
7. Perform accurately the necessary computation.
8. Check the results.

Probably every one here has started more than once to solve a problem before he has read it. Pupils need to be trained to read problems, both to understand the vocabulary and to analyze the problem into separate conditions.

The method I have outlined for solving a mathematical problem is the same that I suggested in reaching a conclusion about buying real estate. In the problems met in daily life, do we first decide upon what we are given or what we want to find? We do not say to ourselves, what do I know? and then try to raise a question that can be answered out of this information. We first have the question. Then we try to find data that will help us answer the question. That is the proper order of thinking in solving a mathematical problem. First, what do I want to find, then what do I need to know to find it. In practical problems we often do not have at hand all of the information needed to solve the problem. Then we try to find it. Now it has been a great limitation on problems in books that they contain just what is needed in the solution of the problem and nothing more. If they are to represent actual conditions as well as possible, there should be problems without enough data and problems containing data not needed in the solution. "An American flag is four feet wide. How wide is each stripe?" That is a better problem for a seventh grade boy than if the number of stripes were given. He has to supply that fact himself and does more thinking of the actual conditions. You have all heard this one.

"A man buys a pair of shoes for seven dollars and gives the clerk a \$10 bill. It was early and the clerk had to go out to a restaurant for change. He came back with the change and gave the customer three dollars and the pair of shoes. About noon the proprietor of the restaurant came in and said "This is a phony bill you gave me." So it was, and the shoe man gave him two good five dollar bills? Who lost, and how much?"

I have given that to classes many times. The answers are interesting. They range from \$3 to \$20. It is a fine example of hiding the main issue with some irrelevant data. This attempt to hide the issue by stirring up dust is not unknown even

in religion, law, and politics. The solution of mathematics problems should require more searching for essentials among non-essentials, if the training is to be as valuable as possible for solving problems that arise outside books.

If a pupil decides correctly upon the processes needed to get the answer, he must, unless he merely makes a happy guess, think the conditions of the problem. That is one of the hardest things to get him to do. He wants a method that works, that can be immediately applied to the problem because of some tag on the problem itself. "A woodcutter gets \$1.50 for sawing a cord of four-foot wood when he saws each stick into two pieces. How much should he get for sawing a cord when he saws each stick into four pieces?" Now you do not get the correct answer to that unless you image what is going on. This training in seeing actual conditions can be begun in the first grade. If the first grade pupil finds himself in situations where he needs to add and subtract to get results that he wants to know, in the second grade he may be able to reconstruct mentally the concrete situation for a problem that is stated for him. I have seen third grade children working problems mentally, thinking them in their heads, and having a good time at it, that some of the college sophomores who were observing them could not work. It was a matter of training to think the conditions of the problems. We have students in algebra, geometry, and even in calculus that can not solve problems because they will not take the trouble to image the conditions.

I believe that algebra problems have been thought useless because they have seemed neither genuine nor important. It is fair to say that if a problem is such that no one in real life would ever solve it or any part of it, it is not a genuine problem. If the answers to a problem would in the ordinary course of events be known before the problem would be formed, we may say that the problem is not genuine. "Twice the width of the Pennsylvania Station in New York exceeds its length by 80 feet. Four times the length exceeds the perimeter by 700 feet. Find the dimensions." This is clearly a case of a problem made up by a person who knew the dimensions of the Pennsylvania Station. Thorndike reports the examination of three textbooks on elementary algebra in which it was found that 57 percent, 42 percent, and 52 percent, respectively, of the problems were of this

"answer known" type. More than half of the verbal problems in these three books then would not be classed as genuine because in ordinary course the answer would have been known before the problem was made.

Thorndike points out that a problem may lack genuineness "because in the real world the situation would probably not occur in the way described or because the answer would not be obtained in the way required." He makes a scale of genuineness from 0 to 10 and suggests as a very charitable criterion that a problem to be called genuine should rank as high as 4 on the scale, and gives five examples of problems that are ranked 4. Here are three of them.

"What angle is five times its complement?"

"A boy knows that his boat can go 6 miles per hour with the current and three miles per hour against the current. How far can he go and return, making the whole trip in just three hours?"

"The diagonal of a rectangle is 102 inches and the base of the rectangle is 3 times the altitude. What is the length of the base?"

These do not represent a high standard of genuineness. Thorndike found that in one of the books examined only about one third of the verbal problems not eliminated as "answer known" graded as high as 4 in genuineness. That is, not 20 percent of the verbal problems equaled in average genuineness the three given above. He next raised the question of importance, that is, is this kind of problem of importance to many people? After testing the remaining 20 percent for importance, he found that of those not more readily solved by arithmetic, or requiring mere substitution in a formula, only one tenth of the problems in the book remained. He publishes that list in his *Psychology of Algebra*. It makes interesting reading. That list tells part of the story of why algebra has been attacked and why it has not been more popular.

Here are some criteria that I think are useful in making problems and selecting problems from lists already made.

1. Let us have as many problems as possible that somebody at some time may conceivably want to solve. Now that requirement is not one of narrow utility. We plow ground because we want to raise corn, not because we want to exercise ourselves or turn over the soil. When we are plowing we may be very much

interested in the activity. It is a real job. We teach factoring because it is worth while to be able to factor certain expressions. While factoring is being studied it may be of great interest. I feel sure that we may find enough material in elementary algebra to last one year that is useful to somebody some time. We have been trying in the last few years to increase the amount of useful material. We have made worth-while progress. Nunn's books are the best source that I know for materials from which problems that may be real to someone at some time may be formed.

2. More algebra problems should require general solutions. "The main service of algebra, as the psychologist sees it, is to teach pupils that we can frame general rules for operating so as to secure the answer to any problem of a certain sort, and express these rules with admirable brevity and clearness by literal symbolism," says Thorndike. We take time and pains to build up habits of computing with general symbols, but make little use of it. Most of the answers to verbal problems are expressed in the particular numbers of arithmetic. The solution of the particular problem should lead to the solution of the general problem. For example:

Particular problem. If A can do a piece of work in 3 days and if B can do the same work in 4 days, how long will it take the two to do the work working together?

General problem. Make a formula for finding how long it will take the two working together to do a piece of work that A can do in x days and B in y days.

Use the formula you have made to find the time taken when they work together if

a. $x = 5, y = 7$.

b. $x = 3 \frac{1}{2}, y = 2 \frac{2}{3}$.

c. $x = 12 \frac{1}{3}, y = 15 \frac{3}{4}$.

This procedure makes apparent the added power that algebra gives.

3. More judgment in the selection of data is required if a number of problems are made to depend upon the same data.

The safe load of a steel beam that is supported at one end and carries a load at the other is given by the formula $W = 6800bd^2/L$, where W is the load in pounds, b the width, d the depth, and L the length of the beam, all of the dimensions being in inches.

A variety of questions may be asked.

What weight will a beam of given dimensions support?

What length must a beam of given width and depth have to support a given load?

What is the effect upon the load of increasing the length, the width and depth remaining the same?

What is the effect upon the load of doubling the width? Of doubling the depth? Of doubling the length?

This formula may be used to give practice in substitution, solving a linear equation, solving a quadratic equation, and in finding the amount of variation and the dependence of variables. A series of problems depending upon one formula is likely to give better training than if each problem introduces a new formula.

4. Problems should be proposed for which not all of the data are given.

5. In the statement of some problems data should be given not needed in the solution. Both criteria 4 and 5 have been illustrated and discussed.

6. Some problems may properly be given that are merely puzzles. These may have great interest for those who "love the game of thought for thought's sake." Many such problems have an added interest because of their history.

It is recognized as good form to repeat the text at the end of the discourse. Elementary algebra should begin and end with the solution of problems. Problems should be used to introduce new meanings and new techniques, to fix them, and to show their use. The solution of problems is my theme and I have not attempted to discuss other elements of the teaching of algebra. There must of course be drill. That drill, if it is well done, may be as interesting as the solution of problems. College entrance examinations and, of last, standard tests have placed too much emphasis upon drill. Pupils have been examined and tested upon elements that can be fixed by drill. The teacher's success is measured by the abilities of pupils to pass these examinations and tests. So the teaching of algebra has been too formal. I have attempted to emphasize the importance of training pupils to put meanings into algebraic symbols and processes. This is most effectively done by solving problems.

THE MIDDLE OF THE ROAD

By W. A. SNYDER,

*Head of Mathematics Department,
New Trier Township High School,
Winnetka, Illinois*

At New Trier the Mathematics Department is rather conservative, and yet, not so conservative that we are not progressive. Then we shall say that we have been pursuing and are now engaged in a program of conservative progressivism. We have not accepted for general use in all of our classes principles and texts that have not been tried and proven to be good nor have we clung to any theories that have been weighed and have been found wanting. We have observed, studied, and experimented with a lot of supposedly new and better ways of presenting secondary mathematics in the past ten to fifteen years and have discovered that, conservatively, about eighty percent of it all is plain bluff, or the desire on the part of some one to get a better hold on a good job, or the natural result of enthusiasm for something new run wild.

For two and one half years we tried out a plan of teaching algebra that was purported to be an individual method. An instructor who was sold on the idea and who started out to prove that this method was much superior to the classroom method was given two classes in algebra each semester, one a slow class and the other a normal class. All kinds of claims were made for this method. It was declared that under this method there were practically no failures. The instructor was very enthusiastic—in fact so much so that common sense and good judgment were discarded. And yet I happen to know that that teacher did not have confidence in his pupils to do as well as pupils in other classes under our regular system of instruction. Very simple but comprehensive standard tests covering a year of algebra were given near the end of each year to the classes in which the individual method was used as well as to the others in algebra and the results were surprisingly disap-

Read before the Conference at University of Illinois, Nov. 19, 1926

pointing and I should have very great fears for the development of mathematicians at our school if such a system ever were adopted. The pupils in these classes stood at the very bottom of the list when the results were tabulated; so much so that we contemplated repetition of the course for these pupils, for they cannot help being handicapped in their junior and senior years of mathematics. I know and have conclusive proof, which I shall not take the time to relate here, that that instructor did not have sufficient confidence in the method to expect his pupils to do as well as other pupils of the same potential abilities, and had been giving credit where credit was not earned according to our standards at New Trier and this accounted for practically no failures. Having discovered through fair tests that the method was not successful, one would think that the experimenter would abide by the results—but too often it is a matter of personal advertising—and too often and too quickly, there are those who are willing to try anything because it is new and perhaps different. A great many educators are claiming too much in the way of results on account of some radical change which they hope to sell to the gullible public.

After all, if twenty percent of all the experimenting in subject matter and teaching were found to be good and worth while, we should thank the experimenter, for that guarantees progressivism, that will keep us out of the so-called "rut."

We do think we have a good mathematics program at New Trier. Yes, we still have some failures; out of 1059 marks given the first month only 42 were F's or less than four percent. Every student understands that the subject cannot be repeated in consecutive semesters more than once. Very few are permitted to take a subject a third time, and these only when they have shown an improved work attitude or have reached a more mature stage in their physical development. Algebra and geometry, one year of each, are required for graduation and all those wishing to enter colleges or universities, where at least one year of mathematics is stipulated, are required to take from one to two semesters of advanced algebra or what we call junior algebra. Our passing marks are A, B, C, D; and F is failure. Two courses in Junior Algebra are given and designated as Algebra IIIA and Algebra IIIB. Algebra IIIA is a good review of first year algebra including more difficult and more

complex exercises, and emphasizing the stated problems in which the student must set up the equations. Radicals, theory of exponents, and quadratics are studied and the exposure is maintained for several weeks. Simple logarithms and determinants of the second and third orders are introduced to add some new work. Algebra IIIB is the more advanced course reviewing briefly the work of the first year course and adding more difficult and special types of factoring, the Factor Theorem, Remainder Theorem, Determinants, Theory of Quadratics, Fractional and Negative Exponents, Logarithms, Progressions, and the Binomial Theorem. Any student who has maintained an average of B or better with no grade below C is permitted to omit Algebra IIIA and enter the classes in Algebra IIIB. All others must elect Algebra IIIA before electing Algebra IIIB. If an Algebra IIIB student who has omitted Algebra IIIA carries the work with an average of A or B, credit will be allowed for IIIA as extra credit. This system has the effect of making the poor or average student of algebra better equipped for more advanced work in mathematics and rewards the better student for a high grade of work. For two years we tried pushing the better students through a year and one half of algebra in one school year and they seemed to get it and understood it at the time, but it was not retained on account of lack of drill work and practice problems and too immature age. We are convinced that it cannot be done so that success in future courses in mathematics can be expected. On account of more work and better work in junior algebra, it is an easier matter for our students to pass the College Board Examinations.

The freshmen are examined before they enter the high school and are classified, according to their ability, industry, and achievement, into *slow*, *normal*, and *fast* groups in algebra. It is best not to let the pupils know that they are in slow, normal, or fast classes; especially it should not be known until the classes are well started. Then if they discover which group they are in, I have found they do not object or complain. If any mistakes are discovered in classification, the shifting should be done the first or second week if possible. All three groups cover about the same topics. The course is enriched for the accelerated groups and a greater number of problems or more difficult exercises are assigned. Of course, we have a majority of A and B

students in the accelerated classes and they are a joy to teach. The course for the slow classes is concentrated, and probably more care is taken in the presentation and explanation of each new topic as we arrive. Fewer problems and exercises are required for home work than in the fast groups but especial care is requested and certain forms and mechanical devices are insisted upon. It is a pleasant surprise to find some leaders developing as we go along. If these slower students were left in normal or fast groups, a much larger number would fail than do fail under our present system. In fact, our failures in algebra are cut over fifty percent. The remaining failing ones practically all pass it on the second attempt.

We have tried a similar scheme in geometry and have found it only partially successful there. This year we have slow classes in geometry but we do not have fast classes. In other words, we take the slowest students in the sophomore class and segregate them into slow classes—approaching the subject in a little different way than with the better students. My experience has been that they get a better start by doing construction exercises for one or two weeks, omitting the proofs. Then take up the simple proofs in the first part of the text slowly at the beginning, giving the students plenty of time to catch the idea of just what a proof consists.

I am not advertising any particular textbook. I believe there is no textbook nor combination of textbooks that will be directly responsible for fewer casualties in mathematics. There are several good textbooks in algebra on the market and any one of these under the guiding hand of a good teacher will help bring the desired results. I shall have something to say later as to the characteristics of a *good* teacher or instructor. Some teachers prefer one textbook to another. Select the preferred one with care. Try it out with several classes for a semester or for a year; if you like it and feel that you would be satisfied with it and your teachers like it, adopt it and don't change too often. A teacher will do better with a text with which he or she is well acquainted. You make your ideas fit the textbook and the text fits well alongside your ideas, and everything works smoothly. The teachers' explanations in the main should agree with and should supplement the explanations given in the textbook. If they do not agree, then it would be better to use a book of exer-

cises, and the teacher do all of the introductory work and analysis in class. This however would go to the opposite extreme from those who would leave all the explaining, grading, and instruction in general to the textbook, without the contact of the teacher—where the teacher acts only as a checker of work completed. I believe neither extreme is good. The teacher needs the help of the text and I never have seen a text that does not need the help of the teacher to get the subject over to the students. The student needs the explanations and illustrations of the text to give him something to which to anchor when his knowledge of the topic is exhausted. The textbook needs the teacher not only to introduce and supplement the materials of the text but also to give the proper inspiration and enthusiasm which any student must have lest the subject become a bore and the result in many cases is failure or an attempt to “get by” through bluffing or cheating or both. I feel sorry for children who miss the opportunity to study under the direction and help of a teacher who has a really big personality and can make the work interesting. A good text with a real teacher is the ideal situation. Anybody can hand out work and check it over when the pupil does it, if he does it at all, but not everybody can teach.

I am wondering if our Normal Schools and Teachers' Colleges and Teacher Training Institutions ever do anything to develop the personality and the character and that fine spirit and cheerful disposition, all of which go toward the makeup of a successful teacher to the extent of about seventy-five percent. The other twenty-five percent must not be overlooked, and that is—an easy mastery of the subject matter which is to be taught. Young men and young women are turned out to teach who have not the requisite qualifications for good and successful teachers and some school boards will elect them because they can be had cheap and they are willing to take a chance. A sympathetic, painstaking, hard-working teacher with a smart and attractive personality and endowed with a spirit of enthusiasm for the success of his pupils to the end that they are inspired with ambition to the mastery of the topic under discussion,—a teacher like that is an asset to any community and is worth two or three times what the average school board is willing to pay. The classroom in algebra is a pleasant meeting between pupils and

teacher, a place where the boys and girls enjoy a forty-five minutes that seem only too short.

Students like friendly competition in the proper atmosphere. Just a few days ago, a mother said to me that her boy, a freshman, had never had the ambition to do his best work under the system of the grade school which he had attended, and that his work had been only average—never to be called excellent—until he entered the high school this fall, where he soon spurted under an atmosphere of competition—he wanted to do well—his work and ability were being compared with that of other boys and girls and he had needed that incentive. At New Trier we grade the students four times in a semester, sending the report cards home each time to be signed by the parents, and this is supplemented by a report showing just where the student stands in comparison with the others in his adviser group which consists of twenty-five to thirty-five boys. The report reads, for example, "twenty were above your boy and six were below him this month. The average for the school is 2.10." The boy's average is checked on the diagram as being 1.25 or perhaps 2.50. In the first case, the boy's parents would know he was below the average of the school, or in the second case, they would understand he was doing a little better than the average. This is one of the devices sponsored by the administration to bring about friendly and wholesome competition, and higher scholarship, and at once to inform the parents of the standing of their children. The parents have a right to know and should be informed. There are many other devices that can be inaugurated by wide-awake teachers to keep students "on their toes."

Too long assignments is often the cause of lack of interest and eventual failure in algebra. Teachers assign too long lessons and kill the spirit of the boys and girls because the home work is drudgery. It is better to err on the side of too little home work assigned than to give too much. I would rather assign a lesson that would take the boy or girl with average ability twenty to thirty minutes than to make the mistake of assigning too much work. The best students will have more freedom and the others will have *some* freedom. Both written and oral drill exercises are necessary to insure the same invariable response to a recurring stimulus, but drill exercises do not enrich the student's notion of some important and controlling principle and are not

intended to do so. I know that students, as we get them from the grade schools, who have been drilled upon the fundamentals in arithmetic, make higher grades on examinations involving thought processes than those who have not been so drilled, the reason being that those skilled in the fundamentals and mechanical processes have mental energy released for the thought processes. I find a great deal of inefficiency in mathematics due to the consumption of energy, drawn off to control physical and mental processes which should be reduced to habit. And of course in the matter of habit, repetition is the big factor. But long lessons are discouraging. Too much practice work insisted upon is a bore and the student will not hold up under it. The conscientious boy or girl will work all the problems assigned and you cannot make those who are not so ambitious do more work by assigning longer lessons. I would not assign more than five or six problems of average difficulty for a lesson nor would I assign more than twenty or twenty-five simple practice exercises. The longer I teach, the more I am convinced that students are driven to just "get by" in their work by teachers who assign too much work and, besides that, fail to make the work one bit interesting. The student who works faithfully and does all the work prescribed and extra work on his own initiative and thereby receives the reward of an "A" grade does not have the freedom that he should have. I mean by that, his work is all outlined for him, he does not have time to read books of his own choice, he has very little, if any, time to give to music, dramatics, manual arts, and such like, and has little or no time for play. Sometimes I think we are filling the child's daily program to overflowing—so many subjects are required that the child has few periods of the day in which the opportunity is afforded for study and too much work has to be carried home to study and prepare after school hours. Therefore, I say we make a mistake if we assign too long lessons with too much home work. Drill work and many practice exercises are necessary in first year algebra but a great deal of this can be done in the class recitation period.

We are studying the problem now at New Trier of handling larger classes as the high school population increases, and a part of the problem in mathematics is in getting enough drill and practice done in the large class. We are trying out a plan involving assistant teachers who help the regular teacher in the

classroom work. For instance, a ten-minute quiz may be given at the beginning of the recitation period involving some algebraic principle or some mechanical processes or both. The assistant can take these papers and upon correcting them can judge, perhaps before the period is over, just about where most of the difficulties occurred and the teacher can repeat the explanation. The experienced teacher might well handle from 50 to 100 accelerated students in beginning algebra with the aid of an assistant who would relieve the regular teacher of much of the checking and grading of papers. These assistants we are getting from the School of Education of Northwestern University and they are being credited with practice work at the University on account of the help they are giving us.

Another change we have made this year in the Mathematics Department is the reduction of the teacher's program from five to four classes by raising the average number in each class. The teachers all like it better and say they do not feel so much worn out at the end of the day. It gives another free period for student conferences or for rest and I think it is a good change and a step in the proper direction.

The matter of grading has probably always been a thorn in the flesh. We have discarded grading on a percentage basis and all papers are graded A, B, C, D, or F; A being the highest mark given and F denoting failure. I ask the teachers in the mathematics department to grade rather leniently the first month and not to give F to any student who is trying even though he may be doing work which would ordinarily warrant an F. Such a mark given the first month will discourage the student generally. Do not expect every student to get the idea perfectly at the beginning and you will not be disappointed. This is probably the case in geometry more often than in algebra. We also grade the student on the way he behaves in the group, his attitude, and his enthusiasm. The question arises "Is he a good citizen in the class organization?" Eventually his final mark in the subject is the teacher's estimate of what he has really accomplished and is then able to do; the final semester grade is not necessarily the average of the several grades given him during the semester.

Frankly I never have been quite sold on fusion mathematics or so-called correlated mathematics. Never have we received a student, transferred from a school where they are using this

system, who seemed to know very much about either algebra or geometry. It has always been a problem to know just where to register them in mathematics. Of course, that alone would not be a fair or conclusive test of the system. For those who will probably never get more than a year or two of mathematics I believe a course in correlated mathematics would be a good thing, something in the nature of a general science course. It could be made up of the very simple processes and principles of algebra, geometry, solid geometry, and trigonometry with practical applications of each in simple problems. I say a course like this might be well to give to those students who do not intend to go beyond the high school, and even then it would be a pity not to expose them to some demonstrative geometry, which is some of the very best training we give in the secondary school curriculum. We still believe in the somewhat old-fashioned mathematics of algebra, geometry, and trigonometry, not sugar-coated to spoil the flavor; concentrated and undiluted. Do the students like it? Yes they do. They like it better than some other subjects in the curriculum.

It is always easy to criticize the school from which a student graduated if he does not do well when he gets into high school, and yet I presume there is some just criticism of the kind of preparation some secondary schools get from the grades; especially is this often true in mathematics. To-night at the regular meeting of the Men's Mathematics Club of the Chicago Area at Chicago, they are to discuss the question, "Shall we discontinue the four operations in common fractions?" and we are told that "if you believe in teaching common fractions, come prepared to defend your belief." I am sorry I cannot be there, I have always thought that they were a necessary evil. I always expect to review fractions in arithmetic with the freshmen, but I do think that freshmen should be able to divide 120 by 20 without writing it down and dividing by long process, and they also ought to be able to multiply 23 by 7 without working it out on paper, and they ought to know how much 7 times 9 is without hesitating, and they ought to know *how* to multiply 76 by 23. Only those skilled in mental arithmetic, the mechanical processes, and the fundamentals, those things which should have become habit through oral and written drill exercises in the grades, can hope to do very good work when they begin their secondary

mathematics. Where this drill has not been done, a great deal of time and energy is wasted in controlling these simple processes which ought to be released for the thought processes necessary for the solution of the problem. I mention this again and emphasize it because it is so essential to good work in secondary mathematics. I would rather have a boy or girl come into the ninth grade skilled in the fundamentals of arithmetic through oral and mental drill than to have that boy or girl come into the ninth grade with a poor background in arithmetic and a smattering of algebra. Those who have had a little algebra will do better than those who have not had it for about one or two weeks, but after that those who are better in mental arithmetic will surpass them. Students should not be entered into the ninth grade when they are too young. We are making a mistake in pushing the child too far ahead in his work. There are many other things in the form of athletics and social training, and real training for citizenship, which those students will miss in high school who are behind their group physically. A boy or girl should be at least 13 years of age to enter the ninth grade and I, personally, would rather have them nearer 14 years. Why push them through? What is the hurry? They will get much more out of high school if they are of the proper physical and social age. I have seen individuals who have been pushed ahead, because of some unproven theory, who have been misfits in their classes all the way through high school. In many cases their parents and their advisers have recommended a fifth year in the high school because they felt that they were too young and too immature to go to college. And it took just that fifth year to bring them out where they might have been for four years had they not been pushed ahead one whole year too far.

Then briefly I have tried to tell you what I think is a conservatively progressive program in mathematics for the ninth, tenth, eleventh, and twelfth years, giving special attention to the ninth year, and have warned against the hurried adoption of certain ideas that are only in the experimental stages, or that are put on the market to sell. Instead of trying to shorten the course in algebra, we have added a semester course for those who are average and below average.

We would separate the freshmen in mathematics into slow, normal, and fast groups.

We would reduce the number of classes to four, especially where much is made of the adviser system and each teacher looks after the interests of a group of twenty-five to thirty-five boys or girls throughout the year. Sympathetic teachers with strong personalities and an easy mastery of the subject are in demand. Nearly always the teacher is responsible for the record of his class. Of course there are exceptions, but the teacher should be able to explain why in any case.

Fifteen to twenty percent failures in mathematics is not necessary. A good school organization with a high standard of work can help reduce this percentage—and I have not seen any method that will surpass the class method in efficiency.

Competition is necessary to produce the best efforts.

Textbooks should be changed not too often. I do not believe in forcing a good teacher to teach a text he does not like. The teacher is more important; don't give him a handicap.

Drill work and practice problems are very necessary but do not overload the students—give freshmen especially some freedom and some time to play. They are only children. Assignments should be short. A grade should be the teacher's estimate of what the student has accomplished and is able to do at the conclusion of any particular period.

Larger classes may be handled by the use of assistants where they are available.

Fusion or correlated mathematics could be made a good practical course for boys and girls who do not wish to continue their education beyond the high school if that fact could be ascertained ahead of time and if it were felt that they would not be able or would not have time to carry the regular course for four years.

Do not push a boy or girl so far ahead that he or she has a physical and social handicap in the group.

NATIONAL COUNCIL PROGRAM

The eighth annual meeting of the National Council of Teachers of Mathematics will be held at Dallas, Texas, February 25 and 26, 1927. The following program will be held:

1. FRIDAY EVENING SESSION, 7:30 P.M. (HILTON HOTEL)

Joint Meeting of the Executive and Local Committees. Important items of business will be discussed at this meeting. Every member of each committee is urged to attend.

2. SATURDAY MORNING SESSION, 9:00 A.M.

(ADOLPHUS HOTEL)

Business Meeting: Reports of the Secretary, Treasurer, Auditing Committee, Editor, President, New Business.

Reports of Classroom Experiments (20 minutes each): "Investigations in the Teaching of Plane Geometry," John R. Clark, Columbia University, New York City; "The Laboratory Method in Teaching Geometry," William A. Austin, Head of Mathematics Department, Venice, California; "Individual versus Group Instruction in Ninth Grade Algebra," C. B. Marquand, West High School, Columbus, Ohio; "Efficiency of Instruction in Large and Small Classes," Leonard D. Haertter, John Burroughs School, Clayton, Missouri.

Discussions (5 minutes each): Dr. Porter, Austin, Texas; Frank Touton, Los Angeles, California; Mary S. Sabin, Denver, Colorado; J. M. Bledsoe, Commerce, Texas.

3. SATURDAY NOON LUNCHEON, 12:30 P.M.

(ADOLPHUS HOTEL)

Luncheon for the Executive and Local Committees.

4. SATURDAY AFTERNOON SESSION, 2:00 P.M.

(ADOLPHUS HOTEL)

"How to Make the Concept of the Locus Real," Elsie Parker Johnson, Oak Park High School, Oak Park, Illinois.

"The Romance of the Number System," H. E. Slaught, University of Chicago, Chicago, Illinois.

"Some of Euclid's Algebra," George W. Evans, Houston, Texas.

5. SATURDAY EVENING BANQUET SESSION, 6:00 P.M.
(ADOLPHUS HOTEL)

Members are requested to send their reservations to Miss Elizabeth Dice, North Dallas High School, Dallas, Texas. The price of the banquet is two dollars and fifty cents a plate.

Greetings from Guests of Honor.

Presentation of the Second Yearbook, Professor W. D. Reeve, Columbia University, New York City.

Discussion.

OFFICERS OF THE NATIONAL COUNCIL OF TEACHERS
OF MATHEMATICS

President, MARIE GUGLE, Assistant Superintendent of Schools, Columbus, O.

Vice-President, W. W. HART, University High School, Madison, Wis.

Secretary-Treasurer, J. A. FOBERG, State Dept. of Education, Harrisburg, Pa.

MEMBERS OF THE EXECUTIVE COMMITTEE

1927—ORPHA WORDEN, Detroit Teachers College, Detroit, Mich.

1927—C. M. AUSTIN, Oak Park High School, Oak Park, Ill.

1928—HARRY ENGLISH, 2907 P St., N. W., Washington, D. C.

1928—HARRY C. BARBER, Charleston High School, Boston, Mass.

1929—W. D. REEVE, Teachers College, Columbia University, New York City.

1929—FRANK C. TOUTON, University of Southern California, Los Angeles, Calif.

EDITOR OF THE MATHEMATICS TEACHER

JOHN R. CLARK, 425 West 123d St., New York City

EDITOR OF THE SECOND YEARBOOK

W. D. REEVE, Teachers College, Columbia University, New York City

CITY AND STATE REPRESENTATIVES

GERTRUDE E. ALLEN, University High School, 58th and Grove St., Oakland, Calif.

MARY S. SABIN, 740 Emerson St., Denver, Colorado.

HARRY ENGLISH, 2907 P St., N. W., Washington, D. C.

INEZ MORRIS, 320 North Seventh St., Terre Haute, Ind.

MARY KELLEY, 446 N. Fountain St., Wichita, Kans.

ANNABEL LEE WHITE, 2446 Maryland Avenue, Baltimore, Md.

ANGELINE WILSON, Grand Rapids, Mich.

J. M. HOWIE, 420 E. 14th St., University Place, Nebr.

W. D. REEVE, 106 Morningside Drive, New York City.

VANCE SMITH, Central High School, Columbus, Ohio.

FLORENCE BROOKS MILLER, 1596 Ansel Road, Cleveland, Ohio.

EDITH E. MORN, 4816 Windsor Ave., Philadelphia, Pa.

ANNA WHITNEY, Yakima High School, Yakima, Wash.

MARY A. POTTER, 827 Lake Avenue, Racine, Wis.

FLORENCE BIXBY, Riverside High School, Milwaukee, Wis.

THE SECOND YEARBOOK
OF THE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Second Yearbook is devoted to Curriculum Problems in Teaching Mathematics. It is divided into three parts as follows: (1) Arithmetic; (2) Junior High School Mathematics; and (3) Senior High School Mathematics.

The contributors are recognized leaders in their fields. Professor F. B. Knight (University of Iowa), Professor G. T. Buswell (University of Chicago), and Miss Jessie P. Haynes (Richmond, Va.) prepared Part I on Arithmetic.

Professor Ralph Beatley (Harvard), Mr. Harry C. Barber (Charlestown High School, Boston), Mr. C. L. Thiele (Assistant Director of Exact Sciences, Detroit), and Professors David Eugene Smith and William D. Reeve (Teachers College) contributed Part II on Junior High School Mathematics.

The curriculum problems of the Senior High School are discussed in Part III by Professor Smith, Miss Gertrude Allen (University High School, Oakland, Calif.), and Mr. E. R. Smith (Headmaster of the Beaver Country Day School, Chestnut Hill, Mass.).

The Yearbook is a distinct contribution to the teaching of mathematics. It was prepared under the direction of a committee, of which Professor W. D. Reeve was Chairman, and is distributed by the Bureau of Publications, Teachers College, New York City, at \$1.25 per copy.